A Propositional Modal Logic of Time Intervals*

Joseph Y. Halpern
IBM Almaden Research Center
San Jose, CA 95120

Yoav Shoham
Stanford University
Computer Science Department
Stanford, CA 94305

October 2, 1996

Abstract: In certain areas of artificial intelligence there is need to represent continuous change and to make statements that are interpreted with respect to time intervals rather than time points. To this end we develop a modal temporal logic based on time intervals, a logic which can be viewed as a generalization of point-based modal temporal logic. We discuss related logics, give an intuitive presentation of the new logic, and define its formal syntax and semantics. We make no assumption about the underlying nature of time, allowing it to be discrete (such as the natural numbers) or continuous (such as the rationals or the reals), linear or branching, complete (such as the reals) or not (such as the rationals). We show, however, that there are formulas in the logic that allow us to distinguish all these situations. We also give a translation of our logic into first-order logic, which allows us to apply some results on first-order logic to our modal one. Finally, we consider the difficulty of validity problems for the logic. This turns out to depend critically, and in surprising ways, on our assumptions about time. For example, if we take our underlying temporal structure to be the rationals, we can show that the validity problem is r.e.-complete, if it is the reals then we can show that validity is \( \Pi^1_1 \)-hard, and if it is the natural numbers, then validity is \( \Pi^1_1 \)-complete.

*Some material in this paper appeared in preliminary form in a paper with the same title in the Proceedings of the First IEEE Symposium on Logic in Computer Science, June 1986. This version is essentially identical to one that appears in *Journal of the ACM* 38:4, 1991, pp. 935–962.
1 Introduction

In at least two areas of Artificial Intelligence, known as “qualitative physics” and “automatic planning”, there is a need for reasoning about continuous processes (such as water filling a slightly leaky container) and having assertions refer to time intervals rather than time points. For example, “the liquid level increased by three inches,” “the robot carried out the task” and “I solved the problem while jogging to the ocean and back” may be true at certain intervals but at no time instant. We are interested in a logic in which such statements can be made, and whose formal semantics mirror the intuitive meaning of these statements.

The machinery developed in theoretical computer science so far is inadequate for our purposes. Modal temporal logic, developed in formal philosophy (e.g., [24]) and first applied to reasoning about programs in [21], interprets formulas over time points. Also, in most formulations, time is assumed to be discrete (the one exception of which we are aware is [4]). A similar comment applies to dynamic logic [22]: formulas are interpreted over time points, and furthermore there is no way to state what happens during the execution of a program.

There have been several extensions of these logics that deal with time intervals rather than just time points. Again, the initial idea of dealing with intervals goes back to the philosophers (e.g., [9], and more recently [12, 26, 5, 31]). In computer science, there has recently been work on process logic [23, 19, 11], where intervals (or “paths”) represent pieces of computation, and even more recently work on interval temporal logic [17, 8]. We review these logics and others in more detail in a later section.

While these interval logics and process logics come closer to satisfying our goals than point-based temporal logics do, they are still not adequate for our needs: they either view time as being discrete, rather than continuous, or they do not provide an adequate set of modal operators for describing the situations we have in mind, or both.

Our logic can be viewed as an extension of point-based modal temporal logic, where we simply replace the notion of satisfaction by a state \( s \models \) by the notion of satisfaction by an interval, or an ordered pair of states \( \langle s, t \rangle \models \). Intuitively, we think of the ordered pair \( (s, t) \) as the closed interval consisting of all points between \( s \) and \( t \). When dealing with only time points, a single accessibility relation and modality are sufficient, since the only relation between non-identical points is “after” (and its inverse “before”). The situation for intervals is slightly more complex, since in addition to “after” we have “immediately after”, “during”, “beginning”, “ending” “overlapping”, etc. It turns out that we can express all twelve relations between two distinct intervals (see [1]) by six modal operators: \( \langle B \rangle \), \( \langle E \rangle \), and \( \langle A \rangle \) (for “begin”, “end” and “after”), and their “transposes” \( \langle B \rangle \), \( \langle E \rangle \), and \( \langle A \rangle \). (In fact, as pointed out by Venema [33], we can express these twelve relations using just \( \langle B \rangle \), \( \langle E \rangle \), and their transposes.) The semantics of these operators is quite natural and simple. For example, \( \langle B \rangle \varphi \) is true of an interval \( \langle s, t \rangle \) exactly if \( \varphi \) is true of some beginning interval \( \langle s, t' \rangle \) with \( t' < t \). Similarly, \( \langle E \rangle \varphi \) is true of an interval \( \langle s, t \rangle \) exactly if \( \varphi \) is true of some interval of which \( \langle s, t \rangle \) is a beginning, that is an interval \( \langle s, t' \rangle \) with \( t' > t \). The definitions of the other two pairs of modal operators is similar.

Although for most of our applications we want to view time as having the structure of the reals, for the sake of generality we give the semantics of these operators with respect to an
arbitrary temporal structure. Thus, time could be discrete (such as the natural numbers) or continuous (such as the rationals or the reals), linear or branching, complete (such as the reals) or not (such as the rationals). Our simple logic has enough expressive power to distinguish these situations.

We also consider the complexity of the validity problem for our logic. When we speak of validity, we do so with respect to a class of temporal structures (where the class could be a singleton consisting just of the reals, the rationals, or the natural numbers). It turns out that the validity problem is surprisingly sensitive to assumptions we make on the structure of time. For example, we can show that in any class of temporal structures that includes at least one with an infinite ascending sequence (i.e., a sequence \( t_1 < t_2 < \ldots \)), the validity problem is at least r.e.-hard. If all the structures in the class are in addition complete (so that every sequence with an upper bound has a least upper bound), then the validity problem becomes \( \Pi_1 \)-hard. As corollaries to these results, we obtain that the validity problem for the rationals is r.e.-hard, while that for the reals or the natural numbers is \( \Pi_1 \)-hard. (The notion of \( \Pi_1 \) is defined formally in Section 8.) We also give some upper bounds for the validity problem, showing, for example, that the validity problem is r.e.-complete for the rationals, and \( \Pi_1 \)-complete for the natural numbers.

The rest of the paper is organized as follows: In the next section, we list a few choices that need to be made when constructing an interval-based logic, and decisions we have made on these issues with regard to our logic. In Section 3 we give the informal syntax and semantics of our logic; we make things formal in Section 4. In Section 5 we show how the logic can be used to capture situations of interest in qualitative physics and automatic planning. In Section 6 we show how formulas of the logic can distinguish various temporal structures. In Section 7 we show that our logic can be translated into a first-order one, allowing us to apply techniques of first-order logic to our logic. In Section 8 we present our results on the difficulty of the validity problem. In Section 9 we review related work on interval logics, both in philosophy and in computer science. We conclude in Section 10 with some interesting open problems.

## 2 Making initial choices

We mentioned in the introduction previous logics of time intervals. In philosophy we have the logics discussed by Hamblin [9], Humberstone [12], Roper [26] and Burgess [5]. In computer science we also have several interval-based logics. Process logic [23, 11] is a generalization of dynamic logic. Interval temporal logic [17, 8] is a generalization of point-based temporal logic. For other related work see [19, 18, 27].

In Section 9 we discuss these logics and their relation to ours in more detail. Here let us just point out issues that distinguish between the different logics, both in philosophy and in computer science.

1. Ontology. Are intervals primitive objects in the logic, or are they defined in terms of points, which are the only primitive objects? In philosophy one finds logic of both kinds. In computer science almost all interval-based logics construct intervals out of points (the
only exception of which we are aware is Allen’s logic [2]). We will join the majority, and construct intervals out of points.

2. Commitment to a particular underlying temporal structure. With no exception, all interval-based temporal logics in computer science have been committed to the discrete and linear view of time. This has not been the case in philosophy. Our logic will be most general in this respect: we will only assume that the set of time points that lie between any two points is totally ordered. This will allow branching and linear time, dense and discrete time, bounded and unbounded time, and so on. Of course, further restrictions that we place on the nature of time induces special properties on the logic, as will become clear when we discuss the complexity of the validity problem for our logic.

3. Choice of tense operators. In computer science, the strong commitment to a discrete and linear order dictated fairly standard modal operators. In philosophy there has been less uniformity. Our logic will be very general in this respect too. We will introduce three very natural pairs of modal operators, which are sufficient to represent all twelve possible relations between two distinct intervals.

4. The relation between the truth-value of a formula over an interval to its truth value over parts of that interval. In computer science, the issue that arises is whether or not locality is assumed; a logic is local if an primitive proposition is true over an interval iff it is true over its starting point. In philosophy, the assumption that is sometimes made is of homogeneity. A logic is homogeneous when, roughly speaking, a proposition is true over an interval iff it is true over all of its subintervals. In our logic we do not assume homogeneity, or any other connection between the truth value of a proposition over an interval to its truth value over any part of that interval.

3 Informal syntax and semantics

Implicit in point-based modal temporal logic is the notion of now, the current instant of time. By way of contrast, in our logic, the key notion is the current interval. Formulas are interpreted over intervals, and we have modal operators that let us refer to other intervals besides the current one. As we mentioned above, we view an interval as an ordered set of points; however, our modal operators will allow us to talk about individual points (or, more accurately, point intervals) as well.

Our logic has more modal operators than one usually encounters in point-based modal temporal logics. This is because there are only two possible relationships between two comparable and distinct points $t$ and $t'$ (namely, that $t$ precedes $t'$ or that $t'$ precedes $t$), while two comparable and distinct intervals can stand in one of twelve different relationships. Specifically, well-formed formulas in our logic will be those of propositional calculus, augmented by the modal operators $\langle A \rangle$, $\langle B \rangle$, $\langle E \rangle$, $\langle A \rangle$, $\langle B \rangle$, and $\langle E \rangle$. Their informal meaning is as follows:

$\langle A \rangle \varphi$: $\varphi$ holds at some interval beginning immediately after the end of the current one
Current interval: |---------------|
⟨A⟩: . |--------| .
⟨B⟩: |--------| .
⟨E⟩: . |---------------|
⟨A⟩: |--------| .
⟨B⟩: |---------------|
⟨E⟩: |---------------|

Figure 1: The six basic modal operators

⟨B⟩φ: φ holds at some interval ending during the current one, beginning when the current one begins

⟨E⟩φ: φ holds at some interval beginning during the current one, ending when the current one ends

⟨A⟩φ: φ holds at some interval ending immediately before the beginning of the current one

⟨E⟩φ: φ holds at some interval of which the current one is a beginning

⟨E⟩φ: φ holds at some interval of which the current one is an end.

Pictorially, the modal operators pick out intervals as shown in Figure 1.

Using these operators we can define some more complex ones:

⟨L⟩φ = def ⟨A⟩⟨A⟩φ: φ holds at some later interval

⟨D⟩φ = def ⟨B⟩⟨E⟩φ ≡ ⟨E⟩⟨B⟩φ: φ holds at some interval during the current one (i.e. φ holds at some proper subinterval of the current interval)

⟨O⟩φ = def ⟨E⟩⟨B⟩φ: φ holds at some “future overlapping” interval.

⟨I⟩, ⟨D⟩ and ⟨O⟩ are similarly defined. Pictorially, these operators select intervals as shown in Figure 2. In fact, these modal operators (A, B, E, L, D, O and their “transposes”) exactly define the twelve possible relations between two distinct intervals (see [1]).

We can define the duals of all these operators as usual: [X]φ ≡ ¬⟨X⟩¬φ (where X is A, B, E, etc.). While ⟨B⟩φ intuitively says that φ is true at some beginning interval, [B]φ says that φ is true at all beginning intervals.

We define both the B and E operators so that they refer to strict subintervals. In particular, [B]φ is vacuously true of point intervals of the form ⟨s, s⟩, since they have no strict beginning intervals. Thus, if we define the formula false to be q ∧ ¬q for some primitive proposition q, we have that the formula [B]false holds precisely of point intervals.

We can use this observation to define a “beginning point” modal operator [[BP]], where [[BP]]φ says that φ holds at the beginning point of the interval:
Current interval: |-----------------|
\langle L \rangle:            |-------|
\langle D \rangle:           |------|
\langle O \rangle:           |------|
\langle L \rangle:           |-------|
\langle D \rangle:           |-----------------|
\langle O \rangle:           |-------------- |

Figure 2: Derived modal operators

\text{[[BP]]} \varphi \equiv ((\varphi \land [B]\text{false}) \lor \langle B \rangle(\varphi \land [B]\text{false}))

By analogy, it is easy to define the “end point” modal operator:

\text{[[EP]]} \varphi \equiv ((\varphi \land [B]\text{false}) \lor \langle E \rangle(\varphi \land [B]\text{false}))

Notice that both “point” operators are their own duals: \text{[[BP]]} \varphi \equiv \neg \text{[[BP]]} \neg \varphi, and similarly for \text{[[EP]]}. To emphasize this fact, we have chosen the single \text{[[]]} notation over the double [ ] and \langle \rangle notation.

As observed by Venema [33], the \langle A \rangle and \langle A \rangle operators are actually definable in terms of the other modal operators. \langle A \rangle is definable by \langle A \rangle \varphi =_{df} \text{[[EP]]} (\langle B \rangle \varphi). \langle A \rangle is similarly definable in terms of \langle B \rangle and \langle E \rangle.

4 Formal syntax and semantics

Syntax. Given a set \Phi_0 of primitive propositions, we form the set of all formulas by closing off under conjunction, negation, and the modal operators discussed above. Thus, if \varphi and \psi are formulas, then so are \neg \varphi, \varphi \land \psi, \langle A \rangle \varphi, \langle B \rangle \varphi, \langle E \rangle \varphi, \langle A \rangle \varphi, \langle B \rangle \varphi and \langle E \rangle \varphi. We use the standard abbreviations: \lor, \exists, and so on.

Semantics. An interpretation is a pair \langle S, V \rangle. \text{S} is a temporal structure \langle T, \leq \rangle, where \text{T} is a set of time points and \leq is a partial order on \text{T}. \text{V} is a function which assigns meaning to the primitive propositions by associating each primitive proposition with the set of intervals where it is true. Thus, \text{V} : \Phi_0 \rightarrow 2^I, where \text{I} = \{\langle t_1, t_2 \rangle : t_1 \leq t_2\}). The only assumptions we will make about \leq is that it has “linear intervals,” which means that for any two points \text{t}_1 and \text{t}_2 such that \text{t}_1 \leq \text{t}_2, the set of points \{t : \text{t}_1 \leq t \leq \text{t}_2\} is totally ordered. In other words, if \text{t}_1 \leq \text{t}_3, \text{t}_1 \leq \text{t}_4, \text{t}_3 \leq \text{t}_2 and \text{t}_4 \leq \text{t}_2, then either \text{t}_3 \leq \text{t}_4 or \text{t}_4 \leq \text{t}_3. Note that given this assumption, the set of points induce a forest-like structure with respect to \leq (a forest is a collection of trees). Actually, no part of the discussion in this paper depends on the assumption. We make it simply because it fits our intuition about the nature of time. In particular, given the assumption about the linearity of intervals, one can intuitively think of the pair \langle t_1, t_2 \rangle as the closed interval of points between \text{t}_1 and \text{t}_2. To investigate the logic in its full generality, we have not imposed any further assumptions on the nature of time, such as linearity or continuity. Of course, we
can easily do so. In fact, as we shall show in the next section, some of these assumptions can essentially be expressed by formulas in the logic.

We interpret formulas over pairs \( \langle t_1, t_2 \rangle \) such that \( t_1, t_2 \in T \) and \( t_1 \leq t_2 \). Given an interpretation \( M \) and an interval \( \langle t_1, t_2 \rangle \), a formula \( \varphi \) is either true in the interval (written \( M, \langle t_1, t_2 \rangle \models \varphi \)) or false (written \( M, \langle t_1, t_2 \rangle \not\models \varphi \)). When clear from the context, the interpretation \( M \) may be omitted, and so in those cases we write simply \( \langle t_1, t_2 \rangle \models \varphi \) and \( \langle t_1, t_2 \rangle \not\models \varphi \).

The truth value of formulas is determined by the semantic rules given below. For convenience, we define the strict (i.e., irreflexive) version of \( \leq \):

\[
t_1 < t_2 \equiv \text{def } t_1 \leq t_2 \wedge \neg(t_2 \leq t_1).
\]

1. For all \( \varphi \in \Phi_0 \), we have \( \langle t_1, t_2 \rangle \models \varphi \) iff \( \langle t_1, t_2 \rangle \in V(\varphi) \).

2. \( \langle t_1, t_2 \rangle \models \neg \varphi \) iff \( \langle t_1, t_2 \rangle \not\models \varphi \)

3. \( \langle t_1, t_2 \rangle \models \varphi_1 \wedge \varphi_2 \) iff \( \langle t_1, t_2 \rangle \models \varphi_1 \) and \( \langle t_1, t_2 \rangle \models \varphi_2 \)

4. \( \langle t_1, t_2 \rangle \models \langle A \rangle \varphi \) iff there exists \( t_3 \) such that \( t_2 < t_3 \) and \( \langle t_2, t_3 \rangle \models \varphi \)

5. \( \langle t_1, t_2 \rangle \models \langle B \rangle \varphi \) iff there exists \( t_3 \) such that \( t_1 \leq t_3, t_3 < t_2 \) and \( \langle t_1, t_3 \rangle \models \varphi \)

6. \( \langle t_1, t_2 \rangle \models \langle E \rangle \varphi \) iff there exists \( t_3 \) such that \( t_1 < t_3, t_3 \leq t_2 \) and \( \langle t_3, t_2 \rangle \models \varphi \)

7. \( \langle t_1, t_2 \rangle \models \langle A \rangle \varphi \) iff there exists \( t_3 \) such that \( t_3 < t_1 \) and \( \langle t_3, t_1 \rangle \models \varphi \)

8. \( \langle t_1, t_2 \rangle \models \langle B \rangle \varphi \) iff there exists \( t_3 \) such that \( t_2 < t_3 \) and \( \langle t_3, t_1 \rangle \models \varphi \)

9. \( \langle t_1, t_2 \rangle \models \langle E \rangle \varphi \) iff there exists \( t_3 \) such that \( t_3 < t_1 \) and \( \langle t_3, t_2 \rangle \models \varphi \)

These definitions induce a meaning on the derived modal operators as well. The reader may verify that \( \langle t_1, t_2 \rangle \models \langle D \rangle \varphi \) iff there exist \( t_3, t_4 \) such that \( t_1 < t_3 < t_4 < t_2 \) and \( \langle t_3, t_4 \rangle \models \varphi \), that \( \langle t_1, t_2 \rangle \models [\langle B \rangle \varphi] \) iff \( \langle t_1, t_1 \rangle \models \varphi \), and so on.

A formula \( \varphi \) is said to be satisfiable with respect to a class of temporal structures \( A \) if, in some interpretation \( \langle \langle T, \langle \rangle \rangle, V \rangle \) such that \( \langle T, \langle \rangle \rangle \in A \), we have \( \langle t_1, t_2 \rangle \models \varphi \) for some \( t_1 \in T \) and \( t_2 \in T \) with \( t_1 \leq t_2 \). A formula is satisfiable in a given temporal structure if it is satisfiable with respect to the singleton consisting of that structure. \( \varphi \) is valid with respect to \( A \) if \( \neg \varphi \) is not satisfiable with respect to \( A \).

Of particular interest to us will be three particular temporal structures, namely, the natural numbers, the rationals, and reals, endowed with the usual ordering relation. We denote these three structures \( N \), \( Q \), and \( R \) respectively.
5 Expressing assertions in the logic

The area of Artificial Intelligence known as qualitative physics is concerned with reasoning about relatively simple physical situations, using only rough and qualitative information, in much the same way as people do in everyday life. Typical problems are predicting the outcome of placing a kettle on a burner, reasoning about liquids flowing between containers, and reasoning about collisions between moving objects. Although reasoning about time is clearly central to qualitative physics, the actual work that has been done makes little use of explicit temporal formalisms (see, e.g., [6]).

To illustrate the fact that our logic lends itself nicely to this research domain, consider representing the sentence “if you open the tap then, unless someone punctures the canteen, the canteen will eventually be filled.” In our logic this assertion is represented by the formula

\[
\text{open-tap} \supset (A)(\neg(D)\text{puncture} \supset [\text{EP}][\text{filled}])
\]

Another area of Artificial Intelligence heavily involved in temporal reasoning is automatic planning, where a (usually simulated) robot must reason about carrying out outstanding tasks, managing available resources, meeting various deadlines and interacting with other agents. Here there has been some use of temporal formalisms, most notably by McDermott [16] and Allen [2]. Our logic is in fact related to their logics; its translation into first-order logic, either as described in a later section or as described in [29], results in a logic that is not unlike those of McDermott and Allen. In [28] we argue, however, that our logic (and its translation into first-order logic) has the advantages of very clear semantics, greater simplicity, and improved flexibility.

To see how our logic lends itself easily to the planning domain, consider the assertion “if the robot executes the charge-battery routine then at the beginning of the following execution of the navigate routine its batteries will be fully charged.” This example is somewhat more complex than the qualitative physics one. We choose it not only because it is typical of statements that come up naturally in the process of planning, but because it illustrates another property of our logic. Our logic is intended as a very basic and fundamental vehicle for representing temporal information. If AI has taught us anything, it is that intelligent information processing relies on a detailed and finely-structured knowledge representation. In this example we demonstrate how more complex definitions can be built on top of our “assembly language” logic.

In [29] we categorize proposition types, showing how we can arrive at coherent definitions of, and distinctions between, what are usually called events, facts, properties, processes, and the like. For example, we define liquid propositions to be those that hold over an intervals iff they hold over all its subintervals (that is, propositions for which the philosophers’ assumption of homogeneity, mentioned in Section 2, holds). For our present purposes it is enough to define the notion of solid propositions. A proposition is said to be solid if no two distinct overlapping intervals ever satisfy it. For example, ‘The robot executed the navigate routine’ is a solid proposition. It is easy to define the notion in the logic:

\[
\text{solid}(\varphi) \text{ def } \varphi \supset \neg(B)\varphi \land \neg(E)\varphi \land \neg(D)\varphi \land \neg(0)\varphi
\]

It is easy to check that \( \varphi \) is a solid proposition in a given temporal structure if solid(\( \varphi \)) holds for every interval in that structure. (Although the formula solid(\( \varphi \)) can be true of an interval
if \( \varphi \) is true of a preceding overlapping interval, this cannot happen if \( \text{solid}(\varphi) \) holds for every interval.)

Assertions of the form “the next time that” are very common, and so it will be useful to define a new binary modal operator. For a solid proposition \( \varphi \), \( [[\text{NTT}]](\varphi, \theta) \) will mean that \( \theta \) holds in the first interval which satisfies \( \varphi \) and which begins after the current interval:

\[
[[\text{NTT}]](\varphi, \theta) =_{def} [A](\theta \supset [A](\varphi \supset \varphi))
\]

Given this definition, the assertion about the robot is simply

\[
\text{charge-battery} \supset [[[\text{NTT}]](\text{navigate}, [[[\text{BP}]]](\text{battery-full}))
\]

### 6 Distinguishing among temporal structures

We have so far deliberately refrained from imposing all but the most elementary constraints on the underlying structure of time. For most applications we will indeed want to add further constraints, such as discreteness, linearity or unboundedness. Interestingly, our logic is sufficiently expressive to capture several such constraints in the logic itself: there are formulas that restrict the class of structures exactly to those satisfying the appropriate constraint. In this section we give several examples of such formulas.

**Discreteness.** A point is *discrete* in a temporal structure if, along any path in the structure which includes that point, the point has a “closest point” on each side (unless it has no points on that side). Formally, we say that a point \( r \) is discrete in a temporal structure \( S \) if for all points \( t \in S \), if \( r < t \) (\( t < r \)) then there exists a point \( s \in S \) such that \( r < s \leq t \) (resp. \( t \leq s < r \)) and such that there does not exist a point \( s' \in S \) with \( r < s' < s \) (resp. \( s < s' < r \)).

A temporal structure is discrete if all points in it are discrete.

Now consider the following formulas:

\[
\begin{align*}
\text{length0} &=_{def} B\text{false} \\
\text{length1} &=_{def} (B\text{true} \land B\neg B\text{false} \\
\text{discrete} &=_{def} \text{length0} \lor \text{length1} \lor (B\text{length1} \land (E\text{length1})
\end{align*}
\]

It was noted earlier that \( B\text{false} \), and therefore also \( \text{length0} \), are true exactly of point intervals. Similarly, \( \text{length1} \) is true of \( \langle s, t \rangle \) exactly if \( s < t \) and there are no points between \( s \) and \( t \). It is easily seen that a temporal structure is discrete exactly if the formula \( \text{discrete} \) is valid in that structure. Thus, \( \text{discrete} \) is valid in \( \mathcal{N} \), but is not even satisfiable by any nonpoint interval in either \( \mathcal{Q} \) or \( \mathcal{R} \).
Density. A temporal structure is dense if between any two comparable points there is a third point, i.e., if \( r < t \) entails that there exists an \( s \) such that \( r < s < t \). Consider the following formula

\[
\text{dense} =_{def} \neg \text{length} 1
\]

Clearly, the formula \text{dense} is valid in a structure \( S \) iff \( S \) is dense. In particular, \text{dense} is valid in \( \mathcal{R} \) and \( \mathcal{Q} \), but not in \( \mathcal{N} \).

Unboundedness. A temporal structure is unbounded if for any point \( s \) there exist points \( r \) and \( t \) such that \( r < s < t \). The following definition:

\[
\text{unbounded} =_{def} \langle A \rangle \text{true} \land \langle \overline{A} \rangle \text{true}
\]

guarantees that \text{unbounded} is valid exactly for the class of unbounded structures.

Linearity. A temporal structure is linear if any two points that are comparable under the symmetric and transitive closure of \( \leq \) are also comparable under \( \leq \); i.e., if there is no branching in the forest induced by the structure (notice that this does not preclude having many “parallel” time lines). Thus, in a linear temporal structure, if two distinct intervals start at the same point, then one must be a prefix of another. Similarly, if two distinct intervals \emph{end} at the same point, then one must be a suffix of another. Consider the following definition:

\[
\text{linear-time} =_{def} \langle A \rangle p \supset [A](p \lor [B]p \lor [\overline{B}]p) \land \\
(\langle \overline{A} \rangle p \supset [\overline{A}](p \lor [\overline{B}]p \lor [B]p)),
\]

where \( p \) is a primitive proposition. It is easy to check that \text{linear-time} captures the notion of a linear temporal structure, in that \text{linear-time} is valid with respect to linear temporal structures, while for any non-linear temporal structure \( S \), there is a valuation \( V \) such that \text{linear-time} is not valid in the interpretation \( \langle S, V \rangle \).

Completeness. It is a standard result of first-order logic that any two dense, linear and unbounded structures are \emph{elementarily equivalent}: they cannot be distinguished by formulas in the first-order logic whose only relation symbols are \('='\) and \( '<' \) (see, e.g., [7]). In particular, it follows that \( \mathcal{Q} \) and \( \mathcal{R} \) are elementarily equivalent. However, as we are about to show, \( \mathcal{Q} \) and \( \mathcal{R} \) are distinguishable in our logic (although, as we shall show in section 7, it is the case that all formulas satisfiable in \( \mathcal{R} \) are also satisfiable in \( \mathcal{Q} \)).

The crucial property that distinguishes \( \mathcal{R} \) from \( \mathcal{Q} \) is that \( \mathcal{R} \) is \emph{complete}: all sequences with an upper bound have a \emph{least} upper bound. \( \mathcal{Q} \) is not complete. For example, an increasing sequence of rationals converging to \( \sqrt{2} \) will not have a least upper bound in \( \mathcal{Q} \). We now give a formula in our logic that distinguishes complete from incomplete temporal structures.

Anticipating some of the constructs that we will need in the next section, let \( p \) and \( \# \) be two primitive propositions and define

\[
\text{less} =_{def} p \land \neg \text{unbounded}
\]

\[
\text{least-upper-bound} =_{def} \text{less} \lor \text{linear-time}
\]

\[
\text{complete} =_{def} \text{least-upper-bound}
\]
cell =_{def} [[\text{BP}]] \# \land [[\text{EP}]] \# \land [D] p \land \langle D \rangle p.

Thus cell is satisfied by an interval exactly if both its begin point and its end point satisfy \# (intuitively, the cell “delimiters”), and if all interior intervals satisfy p (intuitively, the cell “content”), and there is some interior interval satisfying p.

Now consider the following formula:

\text{telescoping} =_{def} \langle B \rangle \text{cell} \land [[\text{EP}]] \neg \# \land [E]([[\text{BP}]] \# \supset \langle B \rangle \text{cell}).

If \langle s,t \rangle \models \text{telescoping} then it is easy to show by induction on k that there exists a sequence s_0, s_1, \ldots, s_k, \ldots such that s = s_0, s_0 < s_1 < \ldots < t, and \langle s_i, s_{i+1} \rangle \models \text{cell}. Thus, we have the following picture:

\begin{array}{ccccccc}
\text{cell} & \text{cell} & \text{cell} & \ldots & t \\
\hline
s_0 & s_1 & s_2 & \ldots \\
\end{array}

In a complete temporal structure, the sequence s_0, s_1, \ldots has a least upper bound, say s'. It is easy to see that \langle s, s' \rangle \models [E](\neg \text{length0} \supset \langle D \rangle \text{cell}). Define

\text{complete} =_{def} \text{telescoping} \supset \langle B \rangle ([E](\neg \text{length0} \supset \langle D \rangle \text{cell}))

The discussion above shows that complete is valid in complete temporal structures. In particular, it is valid for \mathcal{R}. Moreover, \neg \text{complete} is satisfiable in any structure which is not complete. For example, in \mathcal{Q}, consider an infinite sequence of rational numbers s_0, s_1, \ldots converging to \sqrt{2}, with s_0 = 1. Define a valuation \text{V} such that \langle s_i, s_{i+1} \rangle \models \text{cell} and for every point t > \sqrt{2}, \langle t, t \rangle \models \neg \#. It is now easy to check that with respect to this valuation, we have \langle 1, 2 \rangle \models \neg \text{complete}.

7 Translation into first-order logic

Suppose we have a modal logic with Kripke semantics and a formula \varphi in that logic. It is well known that we can find a first-order formula \varphi_t which is satisfiable if \varphi is satisfiable. In fact, if \langle M, s \rangle \models \varphi for some structure M and world s, then we can construct a first-order structure M_t such that M_t \models \varphi_t, where the possible worlds in M are the objects in the domain of M_t and the accessibility relation is made into a binary first-order relation. Although the satisfiability of \varphi implies the satisfiability of \varphi_t, then converse is not true in general. It will hold if the accessibility relations are characterizable in first-order logic. While in many cases of interest the accessibility relation is so characterizable, it is not always the case [31].

We consider a similar of our logic translation into first-order logic. Such a translation was already discussed in [29]; the translation we use here is slightly different, and is a a variant of a translation suggested to us by J. van Benthem. The advantage of using this translation is that it allows us to reduce problems of interest to us here to well-known results in first-order logic. In the previous section we showed how various classes of temporal structures can be distinguished by formulas in the logic. In particular, we showed that there are formulas valid in \mathcal{R} but not
in \( \mathcal{Q} \), and therefore that there are formulas satisfiable in \( \mathcal{Q} \) but not in \( \mathcal{R} \). Using the translation into first-order logic, in this section we show that the converse does not hold: every formula satisfiable in \( \mathcal{R} \) is also satisfiable in \( \mathcal{Q} \). (We will make use of this translation also in the next section, when we discuss upper bounds for the validity problem.)

The translation is straightforward. We use the variable symbols \( t_1, t_2, \ldots \), which will designate time points. The target language is the first-order logic with '=' and '≤', and binary predicate symbols \( p_1, p_2, \ldots \) corresponding to the primitive propositions in the modal logic.

Intuitively, where in the modal logic we would say that a proposition \( p \) was satisfied by the interval \( \langle t_1, t_2 \rangle \) in a certain interpretation, in the first-order logic we will say that the formula \( p(t_1, t_2) \) is true under the appropriate interpretation.

Each modal formula \( \varphi \) is translated into a first-order formula \( \varphi_t \) with two free variables: \( t_1 \) and \( t_2 \). Although the first-order formula will not say this, the reader should think of \( t_1 \) and \( t_2 \) as satisfying \( t_1 ≤ t_2 \) (where \( t_1 ≤ t_2 \) abbreviates \( t_1 ≤ t_2 \) \& \( t_2 = t_1 \)):

1. If \( p \) is a primitive proposition then \( p_t = p(t_1, t_2) \)
2. \( (\neg \varphi)_t = \neg (\varphi_t) \)
3. \( (\varphi \land \varphi')_t = \varphi_t \land \varphi'_t \)
4. \( (\langle B \rangle \varphi)_t = \exists t_3 (t_1 ≤ t_3 \land t_3 ≤ t_2 \land \varphi_t[3/t_2]) \). By \( \varphi_t[3/t_2] \) we mean \( \varphi_t \), with all free occurrences of \( t_2 \) replaced by \( t_3 \).
5. \( (\langle E \rangle \varphi)_t = \exists t_3 (t_1 ≤ t_3 \land t_3 ≤ t_2 \land \varphi_t[3/t_1]) \).
6. \( (\langle A \rangle \varphi)_t = \exists t_3 (t_2 ≤ t_3 \land \varphi_t[2/t_1, 3/t_2]) \).
7. ... and similarly for the other modal operators.

In order to make precise the connection between a modal formula and its first-order counterpart, we define the notion of a faithful first-order interpretation.

**Definition 7.1:** Let \( M = \langle \langle T, \prec \rangle, V \rangle \) be a modal interpretation, and \( M_t = \langle T_t, V_t \rangle \) a first-order interpretation (\( V_t \) is the meaning function that determines the denotation of constant symbols, predicate symbols, and the relation symbol \( \prec \)). We will say that \( M_t \) is faithful to \( M \) if it has the following three properties.

1. \( T = T_t \).
2. \( \leq = V_t(\prec) \).
3. For any primitive proposition \( p \), we have \( V(p) = V_t(p_t) \cap \{ \langle t_1, t_2 \rangle : t_1 ≤ t_2 \} \). Note that we place no constraint on \( V_t \) as far as "reversed intervals" go; if \( t_2 < t_1 \) then we do not care whether \( \langle t_1, t_2 \rangle \in V_t(p_t) \).
Note that all modal interpretations have faithful first-order ones. Conversely, all first-order interpretations in which \( \preceq \) has the right properties (i.e., it is a partial order with linear intervals) are faithful to some modal interpretation.

**Lemma 7.2:** Let \( M = \langle \langle T, \prec \rangle, V \rangle \) be a modal interpretation, \( M_t = \langle T, V_t \rangle \) a first-order interpretation that is faithful to \( M \), \( \varphi \) a modal formula, \( s_1, s_2 \in T \) with \( s_1 \preceq s_2 \), and \( v \) a valuation mapping variables to time points such that \( v(t_1) = s_1 \) and \( v(t_2) = s_2 \). Then \( M, \langle s_1, s_2 \rangle \models \varphi \) iff \( M_t, v \models \varphi_t \).

**Proof:** By a straightforward induction on the structure of \( \varphi \). We consider the case that \( \varphi \) is of the form \( \langle B \rangle \varphi' \) here. Observe that \( M, \langle s_1, s_2 \rangle \models \langle B \rangle \varphi' \) iff \( M, \langle s_1, s_3 \rangle \models \varphi' \) for some \( s_3 \) with \( s_1 \preceq s_3 < s_2 \) iff (by the inductive hypothesis) \( M_t, v' \models \varphi' \), where \( v'(t_1) = s_1 \) and \( v'(t_2) = s_3 \), iff \( M_t, v \models \exists t_3 (t_1 \preceq t_3 \land t_3 \preceq t_2 \land \varphi'(t_3/t_2)) \), where \( v(t_1) = s_1 \) and \( v(t_2) = s_2 \), iff \( M_t, v \models \langle B \varphi' \rangle_t \). This completes the proof in this case; we leave the remaining cases to the reader. □

Clearly, there are first-order interpretations that are not faithful to any modal interpretation. For example, a first-order interpretation need not associate a partial order with \( \preceq \). However, we can exclude such uninteresting interpretations by expressing the appropriate properties of time in a first-order formula. Let \( po \) be the first-order formula saying that \( \preceq \) denotes a partial order, and \( li \) the first-order formula saying that intervals are linear (that is, the set of points that lie between any two points is totally ordered). Recall that these were the only assumptions we made about temporal structures.

\[
po =_{def} \forall t_1, t_2, t_3 (t_1 \preceq t_1 \land ((t_1 \preceq t_2 \land t_2 \preceq t_1) \lor t_1 = t_2) \land (t_1 \preceq t_2 \land t_2 \preceq t_3) \lor t_1 \preceq t_3)
\]

\[
li =_{def} \forall t_1, t_2, t_3, t_4 ((t_1 \preceq t_3 \preceq t_2 \land t_1 \preceq t_4 \preceq t_2) \lor (t_3 \preceq t_4 \lor t_4 \preceq t_3))
\]

\[
ok =_{def} po \land li
\]

These definitions immediately give us:

**Lemma 7.3:** If \( M_t \) is a first-order interpretation such that \( M_t \models ok \), then there is a modal interpretation \( M \) such that \( M_t \) is faithful to \( M \).

**Proposition 7.4:** Let \( \varphi \) be a modal formula, \( A \) a class of temporal structures, and \( \text{time}_A \) a first-order sentence (i.e. a formula with no free variables) whose only relation symbols are \( \preceq \) and \( = \) (that is, it expresses some property of time) and whose class of models is exactly \( A \). Moreover, suppose that \( \text{time}_A \supset ok \) is valid. Then for any first-order interpretation \( M_t \), if \( M_t \models \text{time}_A \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi) \), then there exists a modal interpretation \( M = \langle S, V \rangle \) such that \( M_t \) is faithful to \( M \), \( S \in A \), and for some time points \( s_1, s_2 \) with \( s_1 \preceq s_2 \), we have \( M, \langle s_1, s_2 \rangle \models \varphi \). Conversely, if \( M = \langle S, V \rangle \) is a modal interpretation such that \( S \in A \), and for some time points \( s_1, s_2 \) with \( s_1 \preceq s_2 \), we have \( M, \langle s_1, s_2 \rangle \models \varphi \), and \( M_t \) is a first-order interpretation faithful to \( M \), then \( M_t \models \text{time}_A \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi) \).

**Proof:** Suppose \( M_t \models \text{time}_A \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi) \). Since \( \text{time}_A \supset ok \) is valid, it follows from Lemma 7.3 that there is a modal interpretation \( M \) such that \( M_t \) is faithful to \( M \). By
Lemma 7.2, it now easily follows that there exist time points $s_1, s_2$ in $M$ with $s_1 \leq s_2$ such that there is a modal interpretation $M$ and time points $s_1, s_2$ in $M$ such that $s_1 \leq s_2$ and $M, \langle s_1, s_2 \rangle \models \varphi$. The proof of the converse is similar and left to the reader. $lacksquare$

**Corollary 7.5:** Let $\varphi$ be a modal formula, $\mathcal{A}$ a class of temporal structures, and $\text{time}_\mathcal{A}$ a first-order sentence whose only relation symbols are '$\prec$' and '$=$' and whose class of models is exactly $\mathcal{A}$. Then

1. the first-order formula $\text{time}_\mathcal{A} \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi_1)$ is satisfiable iff the modal formula $\varphi$ is satisfiable with respect to $\mathcal{A}$,

2. the first-order formula $\text{time}_\mathcal{A} \supset \forall t_1, t_2 \ (t_1 \preceq t_2 \supset \varphi_1)$ is valid iff the modal formula $\varphi$ is valid with respect to $\mathcal{A}$.

**Proof:** Again, we just briefly give the idea of the proof. For part 1, suppose $\text{time}_\mathcal{A} \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi_1)$ is satisfiable say in the first-order structure $M_t$. Since $\text{time}_\mathcal{A}$ is a sentence, it follows that $M_t \models \text{time}_\mathcal{A} \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi_1)$. The result now follows from Proposition 7.4. The proof of the converse is similar, as is the proof of part 2. $lacksquare$

We are now ready to apply some known results on first-order logic. First we recall the well-known Löwenheim-Skolem theorem (see, e.g., [7]):

**Theorem 7.6:** (Löwenheim-Skolem) If a first-order formula is satisfiable, then it is satisfiable in a countable interpretation.

**Corollary 7.7:** If a formula in our modal logic is satisfiable in some temporal structure, then it is satisfiable in a countable structure.

**Proof:** Let $\varphi$ be a modal formula. If $\varphi$ is satisfiable in some modal interpretation, then by Corollary 7.5, $\text{ok} \land \exists t_1, t_2 (t_1 \preceq t_2 \land \varphi_1)$ is satisfiable in some first-order interpretation. By Theorem 7.6, this last formula is satisfiable also in some countable interpretation $M_t$. By Proposition 7.4 there exists a modal structure $M$ and points $t_1, t_2$ such that $t_1 \leq t_2$ and $M, \langle t_1, t_2 \rangle \models \varphi$, with $M_t$ faithful to $M$. Finally, we note that by definition, since $M_t$ is countable and faithful to $M$, $M$ too is countable. $lacksquare$

The next theorem is originally due to Cantor, and is proved by the well-known zig-zag argument [7]:

**Theorem 7.8:** Any two countable, linear, dense and unbounded structures are isomorphic with respect to '$\prec$' (i.e., there exists a bijection $f$ between the two structures, such that $t_1 < t_2$ iff $f(t_1) < f(t_2)$).

---

1 Note this would not necessarily be the case if $\text{time}_\mathcal{A}$ were an open formula, say with free variable $x_1,\ldots,x_k$, for then the satisfiability of $\text{time}_\mathcal{A} \land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi_1)$ in $M_t$ would amount to $M_t \models \exists x_1,\ldots,x_k$ ($\text{time}_\mathcal{A}$) $\land \exists t_1, t_2 \ (t_1 \preceq t_2 \land \varphi_1)$, while $M_t \models \text{time}_\mathcal{A} \land \exists t_1, t_2 (t_1 \preceq t_2 \land \varphi_1)$ corresponds to $M_t \models \forall x_1,\ldots,x_k$ ($\text{time}_\mathcal{A}$) $\land \exists t_1, t_2 (t_1 \preceq t_2 \land \varphi_1)$. 

13
Theorem 7.9: A formula in our modal logic is valid with respect to the class of dense, linear
and unbounded structures iff it is valid in the rationals, Q.

Proof: Clearly if a formula is valid with respect to all dense, linear, and unbounded structures,
it is valid in Q. For the converse, it suffices to show that if a formula is satisfiable in some dense,
linear and unbounded structure, then it is also satisfiable in Q. Let φ be a formula and S a
dense, linear and unbounded structure, such that for some modal interpretation \( M = \langle S, \text{V} \rangle \)
and time points \( t_1, t_2 \), we have \( M, \langle t_1, t_2 \rangle \models \phi \). Let \( \phi_i \) be the first-order counterpart of \( \phi \). Also,
let dense1, linear1, and unbounded1 be the three first-order formulas asserting that time is
(respectively) dense, linear and unbounded (their definition is straightforward, and is omitted).
Let \( \text{dlu} =_{\text{def}} \text{dense1} \land \text{linear1} \land \text{unbounded1} \). Note that \( \text{dlu} \supset \text{ok} \). Let \( M_i = \langle S, \text{V}_i \rangle \) be a
first-order interpretation that is faithful to M. By Proposition 7.4, we have \( M_i \models \text{dlu} \land \exists t_1, t_2
( t_1 \leq t_2 \land \phi_i ) \). By Theorem 7.6, there is a countable interpretation \( M'_i \) such that \( M'_i \models \text{dlu} \land \exists t_1, t_2
( t_1 \leq t_2 \land \phi_i ) \). Taking \( M' = \langle S', \text{V}' \rangle \) to be the modal interpretation to which \( M'_i \)
is faithful, by Proposition 7.4 we have that \( M', \langle t_3, t_4 \rangle \models \phi \) for some time points \( t_3 \) and \( t_4 \).
By Theorem 7.8, we have that Q and S' are isomorphic with respect to 'lt'. Let \( \text{V}'' \) be the
valuation that corresponds to \( \text{V}' \) under the isomorphism; thus, for example, if \( f : S \to Q \) is the
isomorphism, we \( \langle t_1, t_2 \rangle \in \text{V}'(p_i) \) iff \( \langle f(t_1), f(t_2) \rangle \in \text{V}''(p_i) \). It is easy to check that \( \langle Q, \text{V}'' \rangle \models \phi \).

Corollary 7.10: If a formula in our modal logic is satisfiable in \( \mathcal{R} \) then it is also satisfiable in
Q.

8 The complexity of the validity problem

We now turn our attention to the complexity of the validity problem for the logic. We begin
with a brief review of the notions of \( \Pi^1_1 \) and its dual \( \Sigma^1_1 \). Further details can be found in [25] or
any other standard textbook of recursive function theory.

Formulas of second-order arithmetic with set variables consist of formulas of first-order arithmetic
(that is, in the language with constant symbols 0 and 1, together with the function symbols
+ and \( \times \)) augmented with expressions of the form \( x \in X \), where \( x \) is a number variable and
\( X \) is a set variable, together with quantification over set variables and number variables.
A sentence is a formula with no free variables. Second-order arithmetic with set variables is a very
powerful language. For example, the following (true) sentence of the language expresses the law
of mathematical induction over the natural numbers:

\[
\forall X (0 \in X \land \forall x ((x \in X \supset x + 1 \in X) \supset \forall x (x \in X)))
\]

A \( \Pi^1_1 \) sentence (resp. \( \Sigma^1_1 \) sentence) of second-order arithmetic with set variables is one of the
form \( \forall X_1 \ldots \forall X_n \varphi \) (resp. \( \exists X_1 \ldots \exists X_n \varphi \)), where \( \varphi \) is a formula of second-order arithmetic with
set variables whose free set variables are among \( X_1, \ldots, X_n \) that has no quantification over set
variables. A set \( A \) of natural numbers is in \( \Pi^1_1 \) (resp. \( \Sigma^1_1 \)) if there is a \( \Pi^1_1 \) sentence (resp. \( \Sigma^1_1 \))
sentence) \( \psi(x) \) with one free number variable \( x \) and no free set variables such that \( a \in A \) iff \( \psi(a) \) holds. \( \Pi^1_1 \) hardness and completeness are defined in the obvious way (the reduction is via one-one recursive functions). It is well-known that \( \Pi^1_1 \)-hard sets are not recursively enumerable (see [25]). In particular, if the validity problem for a class of temporal structures is \( \Pi^1_1 \)-hard, it follows that there can be no complete (recursive) axiomatization for the formulas that are valid with respect to that class of structures.

Later in the paper we will also briefly consider the notion of \( \Pi^2_1 \) and its dual \( \Sigma^2_1 \). In order to define these notions, we need to go to third-order arithmetic, which is the result of taking second-order arithmetic with set variables, and further augmenting to allow expressions of the form \( X \in \mathcal{X} \), where \( X \) is a set variable and \( \mathcal{X} \) is a set of sets variable, together with quantification over set of sets variables (as well as set variables and number variables). Again we take a sentence to be a formula with no free variables. A \( \Pi^2_1 \) sentence (resp. \( \Sigma^2_1 \) sentence) of third-order arithmetic is one of the form \( \forall \mathcal{X}_1 \ldots \forall \mathcal{X}_n \varphi \) (resp. \( \exists \mathcal{X}_1 \ldots \exists \mathcal{X}_n \varphi \)), where \( \varphi \) is a formula of third-order arithmetic that has no quantification over set variables or set of sets variables. The definition of \( A \) being in \( \Pi^2_1 \)- or \( \Sigma^2_1 \)-hard is analogous to that for \( \Pi^1_1 \).

The degree to which the complexity of our logic depends on the underlying temporal structure is striking; depending on the class of temporal structures being considered, the validity problem ranges from being decidable to being \( \Pi^1_1 \)-hard (correspondingly, the satisfiability problem ranges from being decidable to being \( \Sigma^1_1 \)-hard.) Actually, we show that for most interesting classes of temporal structures validity and satisfiability are undecidable. One gets decidability only in very restricted cases, such as when the set of temporal models considered is a finite collection of structures, each consisting of a finite set of natural numbers (since in this case one can simply perform an exhaustive check on all structures). The various hardness properties hold even if we weaken the logic by restricting it to the \( B \), \( E \) and \( A \) operators. We also discuss upper bounds for these problems.

### 8.1 Lower bounds

To make our results precise, we need a few brief definitions. A temporal structure is said to contain an infinitely ascending sequence if it contains an infinite sequence of points \( t_0, t_1, t_2, \ldots \) such that \( t_i < t_{i+1} \). Note that any unbounded structure contains an infinite ascending sequence. A class of temporal structures contains an infinitely ascending sequence if at least one of the structures in it does. We have already defined complete temporal structures; those in which any sequence with an upper bound has a least upper bound. A class of temporal structures is said to be complete if all structures in the class are complete. A class \( \mathcal{A} \) of structures is said to have unboundedly ascending sequences if for any natural number \( n \) there is a structure \( T \in \mathcal{A} \) which contains a sequence \( t_1, t_2, \ldots, t_n \) such that \( t_i < t_{i+1}, 0 < i < n \).

We now state all our lower bound results, and then prove them in detail.

**Theorem 8.1:** The validity problem for any class of temporal structures that contains an infinitely ascending sequence is r.e.-hard.

**Corollary 8.2:** The validity problem for \( \mathcal{N} \), \( \mathcal{Q} \), and \( \mathcal{R} \) is r.e.-hard.
In fact, Theorem 8.1 tells us that the validity problem for almost any interesting class of temporal structures will be r.e.-hard. For example, we have:

**Corollary 8.3:** The validity problem for each of the following classes of temporal structures is r.e.-hard:

1. The class of all temporal structures.
2. The class of all linear temporal structures.
3. The class of all discrete temporal structures.
4. The class of all dense temporal structures.
5. The class of all dense, linear, unbounded temporal structures.

In the case of classes that are complete as well as containing an infinitely ascending sequence, we can show that the validity problem is even harder.

**Theorem 8.4:** The validity problem for complete classes of temporal structures which contain an infinitely ascending sequence is $\Pi^1_2$-hard.

**Corollary 8.5:** The validity problem for $\mathcal{R}$ and for $\mathcal{N}$ is $\Pi^1_1$-hard.

Even for classes of structures which contain no infinite ascending sequence we can often get undecidability results.

**Theorem 8.6:** The validity problem for any complete class of temporal structures which has unboundedly ascending sequences is co-r.e.-hard.

Let $\mathcal{K}$ be the set of temporal structures consisting of the initial segments of the natural numbers, with the usual ordering:

$$
\mathcal{K} = \{ \{ [0..n], \leq \} : n = 0, 1, 2, \ldots \},
$$

$\mathcal{K}$ is useful, for example, when reasoning about possible computations of a program, knowing that the computation is finite but having no bound on its length.

**Corollary 8.7:** The validity problem for $\mathcal{K}$ is co-r.e.-hard.
8.2 Proofs of the lower bounds

The proofs for all these results are quite similar. The idea is to construct formulas that essentially encode the computation of a Turing machine. For Theorem 8.1, we construct a formula that is satisfiable iff the TM started on a blank tape never halts. Since the non-halting problem is co-r.e.-hard, this makes satisfiability co-r.e.-hard, and thus validity r.e.-hard. For Theorem 8.4, we construct a formula that is satisfiable iff there is a computation of the TM that enters the start state infinitely often. For nondeterministic TM’s, this problem is known to be $\Sigma_1^1$-hard \cite{10}, so this gives us that satisfiability is $\Sigma_1^1$-hard, and thus that validity is $\Pi_1^1$-hard. Finally, for Theorem 8.6, we construct a formula that is satisfiable iff the TM halts.

We proceed as follows. Fix a TM $M$ (the construction we are about to describe is independent of whether $M$ is deterministic). We assume without loss of generality that $M$ only writes the symbols 0 and 1. Let $Q$ be the set of $M$’s states, with $q_0$ the unique start state and $q_f$ the unique halting state. We will assume that our language contains all the primitive propositions in the set $L = \{0, 1, *, \#, (q, 0), (q, 1), (q, B): q \in Q\}$, as well as the proposition $\text{corr}$ which we discuss later.

The computation of $M$ started on a blank tape in state $q_0$ is encoded as a sequence of IDs separated by pairs of asterisks: * ID1 ** ID2 ** ID3 ** …. Each ID consists of a sequence of cells. Just as in Section 6, a cell is an interval whose first and last points satisfy $\#$, and whose interior satisfies the “content” of cell, which is one of the elements of $L$. (Thus, we allow for contents other than $p$, which was the only content considered in Section 6.) As usual, 0 means that the content of the cell is 0 and that the head is not pointing at the cell, and likewise for 1. Similarly, $(q, 0)$ means that the content of the cell is 0, that the head is pointing at the cell, and that $M$ is in state $q$. A similar statement holds for $(q, 1)$ and $(q, B)$. $B$ represents the “blank” tape symbol. Thus we have the following slight modification of the definitions of Section 6:

\[
\text{cell}(l) = \text{def} \quad [\text{[BP]#}] \land [\text{[EP]#}] \land [D]l \land \langle D \rangle l \\
\text{cell} = \text{def} \quad \bigvee_{l \in L, \ l \neq \#} \text{cell}(l)
\]

(Note that since the Turing machine is finite then so is $L$, and hence so is the above disjunction.)

An ID is simply an interval delimited by *-cells, with at least one non-* cell in its interior.

\[
\text{ID} = \text{def} \quad [B]\text{cell}(\ast) \land [E]\text{cell}(\ast) \land [D]\text{cell} \land \neg [D]\text{cell}(\ast)
\]

A final (resp. start) ID is an ID such that one of its cells has the head in the final (resp. start) state:

\[
\text{final-ID} = \text{def} \quad \text{ID} \land \langle D \rangle (\text{cell}((q_f, 0)) \lor \text{cell}((q_f, 1)) \lor \text{cell}((q_f, B))) \\
\text{start-ID} = \text{def} \quad \text{ID} \land \langle D \rangle (\text{cell}((q_0, 0)) \lor \text{cell}((q_0, 1)) \lor \text{cell}((q_0, B)))
\]

For convenience, we define a new modal operator $F$.

\[
\langle F \rangle \varphi = \text{def} \quad \langle A \rangle \varphi \lor \langle L \rangle \varphi \\
[F] \varphi = \text{def} \quad \neg \langle F \rangle \neg \varphi
\]
Intuitively, $[\mathcal{F}]\varphi$ says that $\varphi$ holds of all future intervals.

We want to force there to be an infinite sequence of IDs or a finite one ending with a final-ID. This is the job of the following formula:

\[
\text{ID-sequence} =_{\text{def}} [\mathcal{F}](\text{ID} \land \neg \text{final-ID}) \lor \langle A \rangle \text{ID}
\]

**Definition 8.8:** We say that there is a computation starting from $s_0$ if either there exists a finite sequence $s_0, s_1, \ldots, s_k$, $k \geq 1$, such that $s = s_0$, $s_0 < s_1 < \ldots < s_k$, $\langle s_i, s_{i+1} \rangle \models \text{ID}$ for $i < k$ and $\langle s_{k-1}, s_k \rangle \models \text{final-ID}$, or there exists an infinite sequence $s_0 < s_1 < s_2 < \ldots$ and $\langle s_i, s_{i+1} \rangle \models \text{ID}$. 

The following is immediate from the definition:

**Lemma 8.9:** $M, \langle s_0, s_0 \rangle \models \text{ID-sequence}$ iff there is a computation starting from $s_0$.

We next want to write formulas which force the sequence of IDs to encode the computation of the TM $M$ starting on a blank tape. We first need to make sure that the contents of each cell are unique.

\[
\text{unique-val} =_{\text{def}} [\mathcal{F}] (\forall l, l'. \in L, l \neq l' \lor \neg l')
\]

We next want to make sure that the computation starts right and continues right. In order to do this, we need a few preliminary formulas. The formula $2\text{-cell}(x,y)$ holds of an interval in case it consists of two consecutive cells, with respective contents $x$ and $y$. Similarly, $3\text{-cell}(x,y,z)$ holds of an interval just in case the interval consists of three consecutive cells, with respective contents $x$, $y$ and $z$.

\[
2\text{-cell}(x,y) =_{\text{def}} (B)\text{cell}(x) \land (E)\text{cell}(y) \land [D]( [\text{BP}] \land [\text{EP}] ) \supset \text{length0}
\]

\[
3\text{-cell}(x,y,z) =_{\text{def}} (B)\text{cell}(x) \land (E)\text{cell}(z) \land (D)\text{cell}(y) \land [D]( [\text{BP}] \land [\text{EP}] ) \supset (\text{length0} \lor \text{cell}(y))
\]

The following formula now guarantees that the first ID encodes a blank tape with $M$ in the initial state

\[
\text{init-ID} =_{\text{def}} [A]\langle \text{ID} \supset 3\text{-cell}(*,(q_0,B),*) \rangle
\]

Finally, we want to ensure that consecutive IDs obey the rules of the transition function. To that end we will use the proposition $\text{corr}$ (read: “corresponds”), which will be true of an interval iff the interval starts and ends with a cell, and these cells are corresponding cells in consecutive IDs. In the following diagram, each dashed segment represents an interval for which $\text{corr}$ holds:
ID: n n+1 n+2

<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
</tbody>
</table>

......

......

The formulas described below guarantee this property of \textit{corr}. The first formula guarantees that an interval for which \textit{corr} is true starts and ends with a cell:

\[
\text{cell-rule} =_{df} \text{corr} \supset (\langle B \rangle \text{cell} \land \langle E \rangle \text{cell})
\]

Also, an interval that starts with an ID and ends with the * that starts the next ID or ends with an ID and starts with the * that ends the previous ID satisfies \textit{corr}:

\[
\text{ID-rule} =_{df} ((\langle B \rangle \text{ID} \land \langle E \rangle (2\text{-cell}(*,*)) \land \lnot (\langle D \rangle \text{ID}) \supset \text{corr}) \land \\
(\langle \langle E \rangle \text{ID} \land \langle B \rangle(2\text{-cell}(*,*)) \land \lnot (\langle D \rangle \text{ID}) \supset \text{corr})
\]

Next, we stipulate that one \textit{corr} interval may not properly contain another:

\[
\text{not-contains-corr} =_{df} \text{corr} \supset (\lnot \langle B \rangle \text{corr} \land \lnot \langle D \rangle \text{corr} \land \lnot \langle E \rangle \text{corr})
\]

The next formula states that if an interval starts with a cell and ends with an interval satisfying \textit{corr}, then it starts with an interval satisfying \textit{corr}:

\[
\text{corr-starts} =_{df} (\langle B \rangle \text{cell} \land \langle E \rangle \text{corr}) \supset \langle B \rangle \text{corr}
\]

Similarly, if an interval ends with a cell which is not the last cell of an ID, and starts with an interval satisfying \textit{corr}, then it ends with an interval satisfying \textit{corr}. Note that we do not require that the last cell of an ID end a \textit{corr} interval since it may not correspond to a cell in the previous ID. This can happen if the head was in the last cell of the previous ID and moved right.

\[
\text{corr-ends} =_{df} (\langle E \rangle \text{cell} \land \langle A \rangle (\text{cell} \land \lnot \text{cell}(*)) \land \langle B \rangle \text{corr}) \supset \langle E \rangle \text{corr}
\]

Finally, the formula \textit{corr-properties} says that the properties of \textit{corr} described above hold of every future interval:

\[
\text{corr-properties} =_{df} \\
[F(\text{cell-rule} \land \text{ID-rule} \land \text{not-contains-corr} \land \text{corr-starts} \land \text{corr-ends})]
\]
Definition 8.10: We say an interval \( \langle s, t \rangle \) can be subdivided into cells if there exist points \( s_0, s_1, \ldots, s_k \) such that \( s_0 = s, s_k = t, s_0 < \ldots < s_k \), and \( \langle s_i, s_{i+1} \rangle \models \text{cell} \). Note that provided unique-val holds, there is at most one way of subdividing an interval into cells. □

Our next formula guarantees that an interval which starts and ends with a cell can either be subdivided into a finite number of cells or contains an infinite sequence of cells:

\[
\text{subdivide} =_{def} (\langle B \rangle \text{cell} \land \langle E \rangle \text{cell}) \supset [E]([B][P][\#] \supset (\text{cell} \lor \langle B \rangle \text{cell}))
\]

Lemma 8.11: If \( \langle s, t \rangle \models \text{subdivide} \) then either \( \langle s, t \rangle \) can be subdivided into a finite number of cells, or else there is a sequence \( s_0, s_1, \ldots \) and a point \( t' \) such that \( s_0 = s, s_0 < s_1 < \ldots < t' < t \), and \( \langle s_i, s_{i+1} \rangle \models \text{cell} \).

Proof: Since \( \langle s, t \rangle \models \langle B \rangle \text{cell} \land \langle E \rangle \text{cell} \), there must be some \( s_1, t' \) with \( s < s_1 \leq t' < t \) such that \( \langle s, s_1 \rangle \models \text{cell} \) and \( \langle t, t' \rangle \models \text{cell} \). Now we show by induction on \( k \), using \text{subdivide}, that if we have found \( s_0, \ldots, s_k \) with \( s_0 < \ldots < s_k \leq t' \) and \( \langle s_i, s_{i+1} \rangle \models \text{cell} \) for \( i < k \), then either \( s_k = t' \) or there exists \( s_{k+1} \) such that \( s_k < s_{k+1} \) and \( \langle s_k, s_{k+1} \rangle \models \text{cell} \). □

Lemma 8.12 Suppose \( \langle s_0, s_0 \rangle \models \text{corr-properties} \land \text{unique-val} \land \text{init-ID} \land [F] \text{subdivide} \), and suppose that there is a (finite or infinite) computation \( s_0, s_1, s_2, \ldots \) starting from \( s_0 \). Then for each \( i \geq 0 \), the interval \( \langle s_{i+1}, s_{i+2} \rangle \) can be subdivided into a finite number of cells which is the same or one more than the interval \( \langle s_i, s_{i+1} \rangle \). Moreover, the intervals which satisfy \text{corr} and which either start in \( \langle s_i, s_{i+1} \rangle \) or end in \( \langle s_{i+1}, s_{i+2} \rangle \) are exactly those which start with a cell in \( \langle s_i, s_{i+1} \rangle \) and end with the corresponding cell in \( \langle s_{i+1}, s_{i+2} \rangle \).

Proof: The formula init-ID guarantees that the first interval \( \langle s_0, s_1 \rangle \) can be subdivided into exactly three cells. Now suppose that the interval \( \langle s_i, s_{i+1} \rangle \) can be subdivided into a finite number of cells. That is, there exist points \( t_0, \ldots, t_k \) such that \( s_i = t_0 < \ldots < t_k = s_{i+1} \) such that \( \langle t_j, t_{j+1} \rangle \models \text{cell} \) for \( j < k \).

Now look at the interval \( \langle s_{i+1}, s_{i+2} \rangle \). Since it starts with a *-cell and ends with one, from \text{subdivide} we have that it starts with a sequence of cells.

To prove the lemma we show

1. this sequence must be at least \( k \) cells long,
2. among all the intervals that either start with a cell in \( \langle s_i, s_{i+1} \rangle \) or end with one of the first \( k - 1 \) cells in \( \langle s_{i+1}, s_{i+2} \rangle \), corr holds of exactly those that start and end with corresponding cells in the two intervals, and
3. this sequence is at most \( k + 1 \) cells long.
Note that since we have \( \langle s_i, s_{i+1} \rangle \models \text{ID} \) and \( \langle s_{i+1}, s_{i+2} \rangle \models \text{ID} \), it follows that there exists \( t \) and \( u \) with \( s_i < t < s_{i+1} < u < s_{i+2} \) such that \( \langle t, s_{i+2} \rangle \models \text{cell}(*) \) and \( \langle s_{i+1}, u \rangle \models \text{cell}(*) \). By ID-rule, it follows that \( \langle s_i, u \rangle \models \text{corr} \) and \( \langle t, s_{i+2} \rangle \models \text{corr} \); i.e., the interval \( \langle s_i, s_{i+2} \rangle \) can be decomposed into two consecutive subintervals, each satisfying corr. From corr-starts, it follows that each of the \( k \) cells in \( \langle s_i, s_{i+1} \rangle \) starts an interval satisfying corr. These intervals must all end between \( u \) and \( s_{i+2} \); for if they did not, we could obtain a contradiction using not-contains-corr and the fact that both \( \langle s_i, u \rangle \) and \( \langle t, s_{i+2} \rangle \) satisfy corr. Moreover, from not-contains-corr we have that no two distinct intervals satisfying corr can end with the same cell. Therefore there must be at least \( k \) distinct cells in the interval \( \langle s_{i+1}, s_{i+2} \rangle \). From Lemma 8.11, it follows that this interval must actually start with at least \( k \) cells. This proves (1).

For (2), we show by induction on \( j \) that for \( j < k \), the formula corr holds for an interval starting with the \( j \)-th cell in \( \langle s_i, s_{i+1} \rangle \) and ending with the \( j \)-th cell in \( \langle s_i, s_{i+1} \rangle \). For \( j = 1 \), we have already shown that corr holds of the appropriate interval. If \( j > 1 \), we know that the \( j \)-th cell in \( \langle s_i, s_{i+1} \rangle \) starts a corr interval. It cannot end before the \( j \)-th cell in the second ID without violating not-contains-corr. If it ends in a cell after the \( j \)-th cell, then by corr-ends we know that some corr interval ends with the \( j \)-th cell in the second ID. But now not-contains-corr and our inductive hypothesis tell us that this interval can start neither before nor after the \( j \)-th cell in \( \langle s_i, s_{i+1} \rangle \). Thus corr holds for the required intervals. Another application of not-contains-corr shows that there can be no other intervals satisfying corr that either start in \( \langle s_i, s_{i+1} \rangle \) or end with one of the first \( k-1 \) cells in \( \langle s_{i+1}, s_{i+2} \rangle \).

For (3), observe that if the sequence is not either \( k \) or \( k+1 \) cells long, then from Lemma 8.11 it starts with at least \( k+2 \) cells, and the \((k+1)^{st} \) cell does not have contents \(*\). Thus, by corr-ends, there is an interval satisfying corr ending with the \((k+1)^{st} \) cell. Another application of not-contains-corr quickly leads to a contradiction.

We are now in a position to ensure that the computation proceeds according to the transition function of M. Notice that the contents of any three consecutive cells determines the contents of the cell in the next ID which corresponds to the middle cell. Suppose the function \( \delta \) describes this transition, so that if three consecutive cells in an ID are \( i, j \) and \( k \), then \( \delta(i, j, k) \) describes the contents of the cell in the next ID that corresponds to \( j \). (Note that for a nondeterministic TM this function is really a relation). The following formula guarantees that the transitions of M are obeyed at all intervals in the future:

\[
\text{oobeys-}\delta =_{df} [F] (\bigwedge_{i,j,k \in L} ((\text{corr} \land \langle E \rangle 3-\text{cell}(i,j,k)) \cup [A](\text{cell} \cup \text{cell}(\delta(i,j,k)))))
\]

Notice that this is the only formula where the details of the particular TM M play a role.

We can now finally define the formula computation:

\[
\text{computation} =_{df} \text{length0} \land \text{unique-val} \land \text{ID-sequence} \land \text{init-ID} \land [F] \text{subdivide} \land [F] \text{corr-properties} \land \text{oobeys-}\delta
\]
**Definition 8.13**: Suppose \( s_0, s_1, \ldots \) is a computation starting with \( s_0 \). This sequence encodes a computation of \( M \) if each interval \( \langle s_i, s_{i+1} \rangle \) can be subdivided into a finite number of cells, and there is a complete computation \( \text{comp} \) of \( M \) (either infinite or ending with \( M \) in a final state) such that the \( j^{th} \) cell of the interval \( \langle s_i, s_{i+1} \rangle \) is the same as the \( j^{th} \) cell in the \( i^{th} \) ID of \( \text{comp} \).

**Lemma 8.14**: Let \( \langle s, t \rangle \models \text{computation} \). Then there is a computation starting with \( s \). Moreover, any computation starting with \( s \) encodes a computation of \( M \).

**Proof**: Since \( \langle s, t \rangle \models \text{computation} \), in particular \( \langle s, t \rangle \models \text{length} 0 \), so that \( s = t \). By Lemma 8.9 there is a sequence starting with \( s \) which encodes an infinite sequence of IDs. Suppose that \( s_0, s_1, \ldots \) encodes an infinite sequence of IDs, with \( s = s_0 \). We must show that it encodes a computation of \( M \). By Lemma 8.12 each interval encoding an ID can be subdivided into a finite number of cells. Lemma 8.12 also tells us that intervals starting and ending with corresponding cells in consecutive IDs satisfy \( \text{corr} \). The formula \( \text{obeys}-\delta \) is easily seen to guarantee that corresponding cells in consecutive IDs match up right, so that we really are encoding a prefix of a computation of \( M \) in any finite sequence of IDs, and a complete legal computation of \( M \) in any infinite sequence of IDs.

All of the above constitutes the part common to all proofs. The proofs diverge on the punchline. We prove Theorem 8.1 by encoding the nonhalting problem, using the definition of \( \text{final-ID} \) given earlier. If \( M \) is deterministic, then the formula

\[
\text{computation} \land \neg \langle F \rangle \text{final-ID}
\]

is satisfiable in a class of temporal structures containing an infinitely ascending sequence exactly when \( M \) does not halt on a blank tape. This proves Theorem 8.1.

At first it might appear that we could strengthen the result by encoding the halting problem (rather than the nonhalting problem), by considering instead the formula

\[
\text{computation} \land \langle F \rangle \text{final-ID}
\]

Unfortunately, in general this alone will not suffice. Depending on the particular set of temporal structures that are being considered, this formula can be satisfied by “nonstandard” (or, at least, unintended) computations of the TM. For example, if we are considering any dense structure (e.g., \( Q \) or \( \mathcal{R} \)), there is nothing to exclude a structure which satisfies the conjuncts described thus far and which has the following form:

\[
\begin{array}{cccccccc}
| \cdots | \cdots | \cdots | \cdots | \cdots | \cdots & | \cdots \\
\text{ID1} & \text{ID2} & \text{ID3} & \cdots & \text{final-ID}
\end{array}
\]

In other words, we have not precluded models in which the computation is captured by an infinite sequence of intervals which “telescope” to the right, followed by an interval satisfying \( \text{final-ID} \). Although this structure would satisfy the formula \( \text{computation} \land \langle F \rangle \text{final-ID} \), it would not tell us that the Turing machine necessarily halted.
We are able to exclude such nonstandard models of computation in complete structures.

First we add a conjunct that in complete structures eliminates the possibility of an infinite number of cells:

$$
\text{no-telescope} \overset{\text{def}}{=} \neg \langle B \rangle \langle E \rangle \langle D \rangle \text{cell}
$$

**Lemma 8.15:** If $M = \langle S, V \rangle$, $S$ is a complete structure, and $M, \langle s, t \rangle \models \text{no-telescope}$, then there can be no sequence $s_0, s_1, ...$ and point $t'$ such that $s = s_0$, $s_0 < s_1 < ... t' < t$ and $M, \langle s_i, s_{i+1} \rangle \models \text{cell}$.

**Proof:** Suppose there were such a sequence. Let $s'$ be the least upper bound of $s_0, s_1, ...$. (Such as $s'$ exists since $S$ is complete.) Then it is easy to check that $M, \langle s, s' \rangle \models [E] \langle D \rangle \text{cell}$, contradicting our assumption that $M, \langle s, t \rangle \models \text{no-telescope}$. 

Thus, the conjunction $\text{subdivide} \land \text{no-telescope}$ guarantees that any interval (in a complete structure) that starts and ends with a cell can be subdivided into a finite number of cells. We need one more formula which guarantees that if an interval starts and ends with an ID, then it can be subdivided into a finite number of IDs. Define

$$
\text{subdivide-ID} \overset{\text{def}}{=} \langle \text{ID} \supset [A] (\text{cell} \supset \text{cell}(*)) \rangle \land \neg 3-\text{cell}(*,*,*)
$$

Finally, let

$$
\text{standard} \overset{\text{def}}{=} [F](\text{subdivide} \land \text{no-telescope} \land \text{subdivide-ID})
$$

We leave it to the reader to check the following lemma

**Lemma 8.16:** If $M = \langle S, V \rangle$, $S$ is a complete structure, and $M, \langle s, t \rangle \models \text{standard} \land \langle B \rangle \text{ID} \land \langle E \rangle \text{ID}$, then $\langle s, t \rangle$ can be subdivided into a finite number of IDs.

It is now easy to check that in a complete structure, if the formula

$$
\text{computation} \land \text{standard} \land \langle B \rangle \text{start-ID} \land \langle E \rangle \text{final-ID}
$$

is satisfiable then there is a halting computation of the Turing machine $M$. Moreover, if there is a halting computation of the Turing machine $M$, this formula is satisfiable in any complete class of temporal structures which has unboundedly ascending sequences. This proves Theorem 8.6.

Finally, we encode the question of whether the TM returns infinitely often to its start state. Observe that if the formula

$$
\text{computation} \land \text{standard} \land [F](\text{start-ID} \supset (L)\text{start-ID})
$$

is satisfiable then there is a halting computation of the Turing machine $M$. Moreover, if there is a halting computation of the Turing machine $M$, this formula is satisfiable in any complete class of temporal structures which has unboundedly ascending sequences. This proves Theorem 8.6.
is satisfiable in a complete structure, then there is a computation where M returns infinitely often to its start state. Moreover, if M does return infinitely often to its start state, this formula is satisfiable in any complete class of temporal structures which contains an infinite ascending sequence. This, combined with the result mentioned above that the problem of deciding if a nondeterministic TM returns to its start state infinitely often is \( \Sigma^1_1 \) hard, proves Theorem 8.4.

This concludes the proof of our lower bound results.

**Corollary 8.17:** All our hardness results hold even when we weaken the logic to include only the B, E and A operators.

**Proof:** In our constructions we only used these three operators, and ones defined in terms of them. \( \square \)

### 8.3 Upper bounds

We end by briefly discussing upper bounds for the complexity problems. We have the following results.

**Theorem 8.18:**

1. The validity problem for each of the following classes of temporal structures is r.e.-complete:
   
   (a) The class of all temporal structures.
   (b) The class of all linear temporal structures.
   (c) The class of all discrete temporal structures.
   (d) The class of all dense temporal structures.
   (e) The class of all dense, linear, unbounded temporal structures.

2. The validity problem for Q in r.e.-complete.

3. The validity problem for \( N \) is \( \Pi^1_1 \)-complete.

4. The validity problem for \( R \) is in \( \Pi^1_2 \).

5. The validity problem for \( K \) is co-r.e.-complete.

**Proof:** We have already proved all the lower bounds, so it only remains to show the upper bounds.

For (1), we prove the upper bound for dense, linear, unbounded structures. The other proofs are similar. By Corollary 7.5, \( \varphi \) is valid for the class of dense, linear, unbounded structures iff the first-order formula \( \text{d}1u \supset \forall \nu t_1, t_2 \ (t_1 \preceq t_2 \supset \varphi) \) is valid (where \( \text{d}1u \) is the first-order formula discussed in the proof of Theorem 7.9 which which characterizes dense, linear, unbounded
structures). Since validity for first-order logic is well-known to be r.e., the result follows. Note that the proof actually shows that validity with respect to any first-order definable class of structures is r.e.

For (2), note that by Theorem 7.9 validity for \( Q \) is equivalent to validity for dense, linear, unbounded structures, so the result follows immediately from part (1).

For (3), we show that the satisfiability problem for \( \mathcal{N} \) is in \( \Sigma^1_1 \). We do this by showing that given a modal formula \( \varphi \), we can construct a \( \Sigma^1_1 \) sentence \( \psi_{\varphi} \) such that \( \psi_{\varphi} \) is true iff \( \varphi \) is satisfiable in \( \mathcal{N} \). Suppose \( \varphi \) has \( k \) subformulas \( \varphi_1, \ldots, \varphi_k \), where \( \varphi = \varphi_k \). (A subformula of \( \varphi \) is simply a substring of \( \varphi \) which is also a formula.) A pair \( (m, n) \) of natural numbers representing an interval can be encoded by a single number using the pairing function \( f(m, n) = 1/2((m+n)^2 + 3m+n) \). It is well known [25] that \( f \) is a one-one onto map from \( \mathcal{N} \times \mathcal{N} \) to \( \mathcal{N} \). We use the sets \( X_1, \ldots, X_k \) to encode the intervals where the formulas \( \varphi_1, \ldots, \varphi_k \) are true. Thus the formula \( \psi_{\varphi} \) is of the form \( \exists X_1 \ldots X_k \psi' \), where \( \psi' \) is a conjunction encoding some conditions that the sets \( X_i, i = 1, \ldots, k \) must satisfy. For example, if \( \varphi_j \) is of the form \( \varphi_{j_1} \land \varphi_{j_2} \), then one of the conjuncts in \( \psi' \) is:

\[
x \in X_j \equiv (x \in X_{j_1} \land x \in X_{j_2}).
\]

Similarly, if \( \varphi_j \) is of the form \( (B)\varphi_{j'} \), then then (using some obvious abbreviations) we have a conjunct in \( \psi' \) of the form:

\[
x \in X_j \equiv \exists x_1, x_2, x_3, y((x_1 \leq x_2) \land (x_2 \leq x_3) \land (x = f(x_1, x_2)) \land (y = f(x_1, x_3)) \land (y \in X_{j'})).
\]

Finally, we have a conjunction if \( \psi' \) stating that \( X_k \) is nonempty: \( \exists x (x \in X_k) \). We leave it to the reader to check that \( \varphi \) is satisfiable iff \( \psi_{\varphi} \) is satisfiable.

The proof that the satisfiability problem for \( \mathcal{R} \) is in \( \Sigma^1_2 \) proceeds along very similar lines. The only difference is that we can no longer represent an interval of reals \( \langle t_1, t_2 \rangle \) by a single number. Rather, we have to represent it by a set of natural numbers, which encodes two Cauchy sequences of rational numbers (where a rational number \( q \) is represented by a triple \( \langle m_1, m_2, m_3 \rangle \), where \( m_1/m_2 \) encodes the absolute value of \( q \), and \( m_3 \), which is either 0 or 1, encodes its sign). Given a modal formula \( \varphi \), we can construct a \( \Sigma^1_1 \) sentence \( \sigma_{\varphi} \) such that \( \sigma_{\varphi} \) is true iff \( \varphi \) is satisfiable in \( \mathcal{R} \). The sentence \( \sigma_{\varphi} \) is similar in structure to \( \psi_{\varphi} \), except that all the number variables (which were previously used to encode intervals of natural numbers) are replaced by set variables (which encode intervals of reals, as described above), and all the set variables are replaced by set of sets variables. In addition, we need clauses to guarantee that all the set variables used do encode intervals of reals as described above (i.e. to guarantee that they encode a pair of Cauchy sequences). We omit the details here.

Finally, for \( \mathcal{K} \), note that it is obviously decidable if a modal formula \( \varphi \) is satisfiable in a particular finite initial segment of the natural numbers. Since \( \varphi \) is satisfiable in \( \mathcal{K} \) iff it is satisfiable in some initial segment, the satisfiability problem for \( \mathcal{K} \) is clearly r.e. (We just check the initial segments one by one, and report that \( \varphi \) is satisfiable if it is satisfiable in some initial segment.) Thus, the validity problem for \( \mathcal{K} \) is co-r.e. \( \blacksquare \)
9 Related work: interval-based modal logics of time

There is a very rich literature on the algebra of intervals and on interval logics, spanning philosophy and computer science. In AI in particular there have been several important developments in this areas over the past few years. However, none of these have to do with modal interval logics, which are the topic of this comparison section. We will therefore not discuss recent work by Allen and Hayes [3], Ladkin [14], Ladkin and Maddux [15], or Kowalski and Sergot [13]. An concise overview of this and other literature is offered in van Bentheim’s [32].

Although most of the work on modal temporal logic has been point-based, recent years have seen a growing interest in interval-based modal logics. As usual, the initial idea of dealing with intervals goes back to the philosophers (e.g., [9], and more recently [12, 26, 5, 31]). In computer science, there has recently been work on process logic [23, 19, 11], where intervals (or “paths”) represent pieces of computation, and even more recently work on interval temporal logic [17, 8]. The one property all the interval-based logics have in common is that they interpret propositions over intervals of time. They differ among themselves, however, on several counts.

The first distinction is the ontological one mentioned in Section 2: are intervals primitive objects, or is it points that are taken as primitive, with intervals defined in terms of points. In philosophy one finds logics of both kinds. For example, in the logic of Burgess [5], intervals are defined by their end points, whereas in the logics of Humberstone [12] and Roper [26] intervals are primitive objects related by the $\subseteq$ (subinterval) and $<$ (completely before) relations. In computer science all interval-based modal logics construct intervals out of points.

Another distinction between various logics was mentioned earlier and stems from the commitment to a particular underlying temporal structure. With no exception, all interval-based modal temporal logics in computer science have been committed to the discrete linear view of time (see details below). This has not been the case in philosophy. Burgess explicitly assumes a dense linear order. Roper assumes linearity, but apparently nothing beyond that.

Another source of difference between the modal logics is the choice of tense operators. In computer science, the strong commitment to a discrete linear order dictated fairly standard modal operators (see details below). In philosophy there has been less uniformity. For example, the only operator discussed by Humberstone is $F$, standing for “in some interval after the current interval”. The one other operator mentioned by Humberstone as a subject for future research is $R$, standing for “in some interval immediately after the current interval.” Roper uses two other modal operators, which are also adopted by Burgess: $G$ (for “in all intervals beginning during the current interval”), and $H$ (for “in all intervals ending in the current interval”). Clearly all these operators are easily definable in our logic.

For a more detailed discussion of temporal logics in philosophy, both point-based and interval-based, see [31]. We now discuss particular interval-based logics in computer science. SOAPL is a fairly complex logic introduced by Parikh [19]. It has two kinds of formulas: those interpreted over states and those interpreted over “paths”, or sequences of states. Parikh proved that validity in the logic is (nonelementarily) decidable, but did not provide a complete deductive system. Nishimura’s logic [18] is an attempt to merge temporal logic with dynamic logic. He showed his logic to be as expressive as SOAPL. His logic too is rather complex, and maintains the distinction between “state” formulas and “path” formulas.
At roughly the same time Pratt introduced process logic, in which formulas are interpreted over paths [23]. Later Harel, Kozen and Parikh refined the formulation [11]. They introduced two new modal operators (in addition to the ones introduced by dynamic logic): \( f \) (first) and \( \text{suf} \) (roughly, until). More precisely,

\[
\langle s_1, \ldots, s_n \rangle \models f \varphi \text{ iff } \langle s_1 \rangle \models \varphi, \text{ and }
\]

\[
\langle s_1, \ldots, s_n \rangle \models \varphi \text{ suf } \psi \text{ iff, for some } j, \langle s_j, \ldots, s_n \rangle \models \psi, \text{ and for all } i \text{ such that } 1 \leq i < j,
\]

\[
\langle s_i, \ldots, s_n \rangle \models \psi.
\]

Harel et al. show that satisfiability in the resulting logic is decidable (although not necessarily elementarily so), and give a complete axiomatization for it. Those results depend on assuming that a primitive proposition is true over an interval iff it is true at the first time point of that interval; that is, assuming locality. This property is captured in process logic by the axiom schema \( p \equiv fp \), for a primitive proposition \( p \). The assumption of locality is really at odds with the reason for our interest in a logic of intervals, since, as Harel et al. themselves put it, “every path formula ultimately expresses properties of states.” They mention the fact that without the axiom \( p \equiv fp \) there are path properties that cannot be expressed, and leave open the question of decidability in the absence of this axiom of locality. Later Streett settled this question by showing that global propositional process logic is \( \Pi^1_1 \)-complete [30].

Interval temporal logic [8, 17] is also an extension of temporal logic in which formulas are interpreted over paths. The two modal operators considered there are \( \bigcirc \) (next) and ; (chop). The meaning of these operators is given by:

\[
\langle s_1, \ldots, s_n \rangle \models \varphi ; \psi \text{ iff } \langle s_1, \ldots, s_i \rangle \models \varphi \text{ and } \langle s_i, \ldots, s_n \rangle \models \psi \text{ for some } i
\]

\[
\langle s_1, \ldots, s_n \rangle \models \bigcirc \varphi \text{ iff } \langle s_2, \ldots, s_n \rangle \models \varphi.
\]

Thus, the \( \bigcirc \) operator strongly commits ITL to the discrete view of time. In [8] it is shown that satisfiability for ITL is undecidable, and that if locality is assumed then satisfiability is decidable but nonelementary. The ITL extension of temporal logic is different from ours in two ways. First, in our logic we are not committed to viewing time as discrete. Second, even if we assume discreteness of time in our logic, the two logics are not comparable in their expressive power. On the one hand the chop operator of ITL is not definable in our logic (this is proved formally by Venema in [33]), and on the other hand we provide means of referring to intervals outside the reference interval, which ITL does not.

Schwartz, Melliar-Smith, and Vogt [27] offer another interval-based temporal logic. They augment modal temporal logic by constructs referring to intervals explicitly: if \( \varphi \) is a formula then so is \( [I] \varphi \), where \( I \) is an interval designator. For example, the formula \([ (x=y) \Rightarrow (y=16) ] \square (x>z)\) is intended to mean that \( x \) is greater than \( z \) throughout the interval beginning at the first time \( x \) equals \( y \) and ending at the first time after that when \( y \) equals 16. Intervals are assumed to consist of linearly ordered and discrete time points, and again locality is assumed. In [20] it is shown that satisfiability for this logic is nonelementarily decidable.

Finally, Venema has recently obtained further results regarding our logic [33]. Besides the expressibility results mentioned above, he proves that all the classes of structures which were
shown in Theorem 8.18 to have a validity problem that is r.e. complete, actually have a relatively
elegant complete axiomatization. In particular, he shows that there is a complete axiomatization
for \( Q \).

10 Conclusions

We presented a new interval logic which generalizes point-based temporal logic. The syntax and
semantics are very simple, and the logic allows one to express naturally statements that refer to
intervals of time and to continuous processes.

We showed that the logic is expressive enough to identify several classes of temporal struc-
tures, such as the classes of dense structures, linear structures, and complete structures. At the
same time, we showed that some classes of structures cannot be distinguished in the logic, one
example being the class of dense, linear and unbounded structures and the singleton consisting
of the rationals \( Q \).

We gave several results on the complexity of the validity problem for the logic. For all but
the simplest classes of temporal structures, validity is undecidable. For classes of structures
with infinitely ascending sequences, such as the rationals \( Q \), validity is r.e.-hard. For classes of
structures which contain unboundedly ascending sequences, validity is co-r.e.-hard. For complete
classes of structures with infinitely ascending sequences, such as \( N \) or \( R \), satisfiability is \( \Pi_1 \)-hard.
Notice that the \( \Pi_1 \)-hardness and co-r.e.-hardness results imply nonaxiomatizability.

Finally, we gave several upper bounds for the validity problem. For \( N, Q \) and \( K \), we showed
that the upper bounds match the lower ones. For \( R \) we gave a less tight upper-bound.

It is surprising that such a natural logic of time has never been explored before. Many
fascinating open problems still remain, and they include the following:

1. Can we find matching upper and lower bounds for the validity problem with respect to
\( R \)?

2. What results can we get for other natural classes of temporal structures?

3. What happens to the complexity of the validity problem if we slightly modify the logic?
We have already remarked that our lower bounds hold even if we restrict the logic to
the \( B, E \) and \( A \) operators, but we do not know what happens for weaker or incomparable
combinations of modal operators, for example the set \( \{ D, \overline{D} \} \) or the set \( \{ B, E \} \).

4. The motivation for our logic was the need to reason about situations of interest in Artificial
Intelligence. Are the hardness results for the validity problem a sign of failure? We think
not. Our logic is very natural, and the meaning of the various operators is quite intuitive.
The fact that an efficient general-purpose theorem prover for the logic is unattainable will
hardly come as a shock to anyone in AI. What we need to do, now that we have a natural
and expressive logic, is to identify classes of formulas about which reasoning is easier than
in the general case.
Acknowledgements. We thank Dana Angluin, Mike Fischer, Yoram Moses, Moshe Vardi, Yde Venema, and especially Johan van Benthem for their comments.
References


