Propositional Calculus E

Inference rules

(3.1) **Leibniz:** $\dfrac{P = Q}{E[V{:=}P] = E[V{:=}Q]}$ (V a metavariable)

(3.2) **Transitivity:** $\dfrac{P = Q,\ Q = R}{P = R}$

(3.3) **Equanimity:** $\dfrac{P,\ P \equiv Q}{Q}$

Derived inference rules

(3.11) **Redundant** $true$**:** $\dfrac{P}{P \equiv true}$ $\dfrac{P \equiv true}{P}$

(3.91) **Modus ponens:** $\dfrac{P,\ P \Rightarrow Q}{Q}$

(4.8) **Monotonicity:** $\dfrac{P \Rightarrow Q}{E[V{:=}P] \Rightarrow E[V{:=}Q]}$ (for parity of V even in E)

(4.9) **Antimonotonicity:** $\dfrac{P \Rightarrow Q}{E[V{:=}P] \Leftarrow E[V{:=}Q]}$ (for parity of V odd in E)

Equivalence and true

(3.5) **Axiom, Associativity of** $\equiv$**:** $((P \equiv Q) \equiv R) \equiv (P \equiv (Q \equiv R))$

(3.6) **Axiom, Symmetry of** $\equiv$**:** $P \equiv Q \equiv Q \equiv P$

(3.7) **Axiom, Identity of** $\equiv$**:** $true \equiv Q \equiv Q$

(3.8) **Reflexivity of** $\equiv$**:** $P \equiv P$

(3.9) $true$

(3.12) **Metatheorem:** Any two theorems are equivalent.

Negation, inequivalence, and false

(3.15) **Axiom, Definition of** $false$**:** $false \equiv \neg true$

(3.17) **Axiom, Distributivity of** $\neg$ **over** $\equiv$**:** $\neg(P \equiv Q) \equiv \neg P \equiv Q$

(3.14) **Axiom, Definition of** $\not\equiv$**:** $(P \not\equiv Q) \equiv \neg(P \equiv Q)$

(3.18) $\neg P \equiv Q \equiv P \equiv \neg Q$

(3.19) **Double negation:** $\neg\neg P \equiv P$

(3.16) **Negation of** $false$**:** $\neg false \equiv true$

(3.20) $(P \not\equiv Q) \equiv \neg P \equiv Q$

(3.13) $\neg P \equiv P \equiv false$

(3.21) **Symmetry of** $\not\equiv$**:** $(P \not\equiv Q) \equiv (Q \not\equiv P)$

(3.22) **Associativity of** $\not\equiv$**:** $((P \not\equiv Q) \not\equiv R) \equiv (P \not\equiv (Q \not\equiv R))$

(3.23) **Mutual associativity:** $((P \not\equiv Q) \equiv R) \equiv (P \not\equiv (Q \equiv R))$

(3.24) **Mutual interchangeability:** $P \not\equiv Q \equiv R \;\equiv\; P \equiv Q \not\equiv R$

Disjunction

(3.29) **Axiom, Symmetry of $\vee$:** $P \vee Q \;\equiv\; Q \vee P$

(3.30) **Axiom, Associativity of $\vee$:** $(P \vee Q) \vee R \;\equiv\; P \vee (Q \vee R)$

(3.31) **Axiom, Idempotency of $\vee$:** $P \vee P \;\equiv\; P$

(3.32) **Axiom, Distributivity of $\vee$ over $\equiv$:**
$P \vee (Q \equiv R) \;\equiv\; P \vee Q \;\equiv\; P \vee R$

(3.33) **Axiom, Excluded Middle:** $P \vee \neg P \;\equiv\; \mathit{true}$

(3.34) **Zero of $\vee$:** $P \vee \mathit{true} \;\equiv\; \mathit{true}$

(3.35) **Identity of $\vee$:** $P \vee \mathit{false} \;\equiv\; P$

(3.36) **Distributivity of $\vee$ over $\vee$:** $P \vee (Q \vee R) \;\equiv\; (P \vee Q) \vee (P \vee R)$

(3.37) $P \vee Q \;\equiv\; P \vee \neg Q \;\equiv\; P$

Conjunction

(3.40) **Axiom, Golden rule** : $P \wedge Q \;\equiv\; P \;\equiv\; Q \;\equiv\; P \vee Q$

(3.41) **Symmetry of $\wedge$:** $P \wedge Q \;\equiv\; Q \wedge P$

(3.42) **Associativity of $\wedge$:** $(P \wedge Q) \wedge R \;\equiv\; P \wedge (Q \wedge R)$

(3.43) **Idempotency of $\wedge$:** $P \wedge P \;\equiv\; P$

(3.44) **Identity of $\wedge$:** $P \wedge \mathit{true} \;\equiv\; P$

(3.45) **Zero of $\wedge$:** $P \wedge \mathit{false} \;\equiv\; \mathit{false}$

(3.46) **Distributivity of $\wedge$ over $\wedge$:** $P \wedge (Q \wedge R) \;\equiv\; (P \wedge Q) \wedge (P \wedge R)$

(3.47) **Contradiction:** $P \wedge \neg P \;\equiv\; \mathit{false}$

(3.48) **Absorption:** (a) $P \wedge (P \vee Q) \;\equiv\; P$
(b) $P \vee (P \wedge Q) \;\equiv\; P$

(3.49) **Absorption:** (a) $P \wedge (\neg P \vee Q) \;\equiv\; P \wedge Q$
(b) $P \vee (\neg P \wedge Q) \;\equiv\; P \vee Q$

(3.50) **Distributivity of $\vee$ over $\wedge$:** $P \vee (Q \wedge R) \;\equiv\; (P \vee Q) \wedge (P \vee R)$

(3.51) **Distributivity of $\wedge$ over $\vee$:** $P \wedge (Q \vee R) \;\equiv\; (P \wedge Q) \vee (P \wedge R)$

(3.52) **De Morgan:** (a) $\neg(P \wedge Q) \;\equiv\; \neg P \vee \neg Q$
(b) $\neg(P \vee Q) \;\equiv\; \neg P \wedge \neg Q$

(3.53) $P \wedge Q \;\equiv\; P \wedge \neg Q \;\equiv\; \neg P$

(3.54) $P \wedge (Q \equiv R) \;\equiv\; P \wedge Q \;\equiv\; P \wedge R \;\equiv\; P$

(3.55) $P \wedge (Q \equiv P) \;\equiv\; P \wedge Q$

(3.56) **Replacement:** $(P \equiv Q) \wedge (R \equiv P) \;\equiv\; (P \equiv Q) \wedge (R \equiv Q)$

(3.57) **Definition of $\equiv$:** $P \;\equiv\; Q \;\equiv\; (P \wedge Q) \vee (\neg P \wedge \neg Q)$

(3.58) **Exclusive or:** $P \;\not\equiv\; Q \;\equiv\; (\neg P \wedge Q) \vee (P \wedge \neg Q)$

(3.62) $(P \wedge Q) \wedge R \equiv P \equiv Q \equiv R \equiv P \vee Q \equiv Q \vee R \equiv R \vee P \equiv P \vee Q \vee R$

Implication

(3.65) **Axiom, Implication:** $P \Rightarrow Q \equiv P \vee Q \equiv Q$

(3.64) **Axiom, Consequence:** $P \Leftarrow Q \equiv Q \Rightarrow P$

(3.66) **Implication:** $P \Rightarrow Q \equiv P \wedge Q \equiv P$

(3.63) **Implication:** $P \Rightarrow Q \equiv \neg P \vee Q$

(3.67) **Contrapositive:** $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$

(3.68) $P \Rightarrow (Q \equiv R) \equiv P \wedge Q \equiv P \wedge R$

(3.69) **Distributivity of** $\Rightarrow$
(a) **over** $\equiv$**:** $P \Rightarrow (Q \equiv R) \equiv P \Rightarrow Q \equiv P \Rightarrow R$
(b) **over** $\vee$**:** $P \Rightarrow Q \vee R \equiv (P \Rightarrow Q) \vee (P \Rightarrow R)$
(c) **over** $\wedge$**:** $P \Rightarrow (Q \wedge R) \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R)$
(d) **over** $\Rightarrow$**:** $P \Rightarrow (Q \Rightarrow R) \equiv (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$

(3.70) **Shunting:** $P \wedge Q \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$

(3.71) $P \Rightarrow (Q \Rightarrow P)$

(3.72) $P \wedge (P \Rightarrow Q) \equiv P \wedge Q$

(3.73) $P \wedge (Q \Rightarrow P) \equiv P$

(3.74) $P \vee (P \Rightarrow Q) \equiv true$

(3.75) $P \vee (Q \Rightarrow P) \equiv Q \Rightarrow P$

(3.76) $P \vee Q \Rightarrow P \wedge Q \equiv P \equiv Q$

(3.77) **Reflexivity of** $\Rightarrow$**:** $P \Rightarrow P \equiv true$

(3.78) **Right zero of** $\Rightarrow$**:** $P \Rightarrow true \equiv true$

(3.79) **Left identity of** $\Rightarrow$**:** $true \Rightarrow P \equiv P$

(3.80) $P \Rightarrow false \equiv \neg P$

(3.81) $false \Rightarrow P \equiv true$

(3.82) **Weakening/strengthening:** (a) $P \Rightarrow P \vee Q$
(b) $P \wedge Q \Rightarrow P$
(c) $P \wedge Q \Rightarrow P \vee Q$
(d) $P \vee (Q \wedge R) \Rightarrow P \vee Q$
(e) $P \wedge Q \Rightarrow P \wedge (Q \vee R)$

(3.83) **Modus ponens:** $P \wedge (P \Rightarrow Q) \Rightarrow Q$

(3.84) $(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv (P \vee Q \Rightarrow R)$

(3.85) $(P \Rightarrow R) \wedge (\neg P \Rightarrow R) \equiv R$

(3.86) **Mutual implication:** $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \equiv (P \equiv Q)$

(3.87) **Antisymmetry:** $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \equiv Q)$

(3.88) **Transitivity of** $\equiv$**:** $(P \equiv Q) \wedge (Q \equiv R) \Rightarrow (P \equiv R)$

(3.89) **Transitivity:** (a) $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$

(b) $(P \equiv Q) \wedge (Q \Rightarrow R) \;\Rightarrow\; (P \Rightarrow R)$

(c) $(P \Rightarrow Q) \wedge (Q \equiv R) \;\Rightarrow\; (P \Rightarrow R)$

(??) **Metatheorem Modus ponens:**
If P and $P \Rightarrow Q$ are theorems, so is Q

Leibniz as an axiom

(3.93) **Axiom, Leibniz:** $X = Y \;\Rightarrow\; E[V{:=}X] = E[V{:=}Y]$

(3.94) **Substitution:** (a) $X = Y \wedge E[V{:=}X] \;\equiv\; X = Y \wedge E[V{:=}Y]$

(b) $X = Y \;\Rightarrow\; E[V{:=}X] \;\equiv\; X = Y \;\Rightarrow\; E[V{:=}Y]$

(c) $Q \wedge X = Y \;\Rightarrow\; E[V{:=}X] \;\equiv\; Q \wedge X = Y \;\Rightarrow\;$
$E[V{:=}Y]$

(3.95) **Replace by** true**:** (a) $P \Rightarrow E[V{:=}P] \;\equiv\; P \Rightarrow E[V{:=}\mathit{true}]$

(b) $Q \wedge P \;\Rightarrow\; E[V{:=}P] \;\equiv\; Q \wedge P \;\Rightarrow\; E[V{:=}\mathit{true}]$

(3.96) **Replace by** false**:** (a) $E[V{:=}P] \Rightarrow P \;\equiv\; E[V{:=}\mathit{false}] \Rightarrow P$

(b) $E[V{:=}P] \Rightarrow P \vee Q \;\equiv\; E[V{:=}\mathit{false}] \Rightarrow P \vee Q$

(3.97) **Replace by** true**:** $P \wedge E[V{:=}P] \;\equiv\; P \wedge E[V{:=}\mathit{true}]$

(3.98) **Replace by** false**:** $P \vee E[V{:=}P] \;\equiv\; P \vee E[V{:=}\mathit{false}]$

(3.99) **Boole:** $E[V{:=}P] \;\equiv\; (P \wedge E[V{:=}\mathit{true}]) \vee (\neg P \wedge E[V{:=}\mathit{false}])$

Monotonicity theorems

(4.1) **Monotonic** $\vee$**:** $(P \Rightarrow Q) \;\Rightarrow\; (P \vee R \;\Rightarrow\; Q \vee R)$

(4.2) **Monotonic** $\wedge$**:** $(P \Rightarrow Q) \;\Rightarrow\; (P \wedge R \;\Rightarrow\; Q \wedge R)$

(4.3) **Monotonic consequent:** $(P \Rightarrow Q) \;\Rightarrow\; ((R \Rightarrow P) \;\Rightarrow\; (R \Rightarrow Q))$

(4.4) **Antimonotonic** $\neg$**:** $(P \Rightarrow Q) \;\Rightarrow\; (\neg P \Leftarrow \neg Q)$

(4.5) **Antimonotonic antecedent:** $(P \Rightarrow Q) \;\Rightarrow\; ((P \Rightarrow R) \;\Leftarrow\; (Q \Rightarrow R))$

(4.7) **Metatheorem Montonicity:**
Provided $P \Rightarrow Q$ is a theorem and the parity of the position of V in E is even, $E[V{:=}P] \;\Rightarrow\; E[V{:=}Q]$.
Provided $P \Rightarrow Q$ is a theorem and the parity of the position of V E is odd, $E[V{:=}P] \;\Leftarrow\; E[V{:=}Q]$.

(4.11) $(P \Rightarrow P') \;\Rightarrow\; ((Q \Rightarrow Q') \;\Rightarrow\; (P \wedge Q \;\Rightarrow\; P' \wedge Q'))$

(4.12) $(P \vee Q \vee R) \wedge (P \Rightarrow S) \wedge (Q \Rightarrow S) \wedge (R \Rightarrow S) \;\Rightarrow\; S$

Proof techniques and strategies

(3.10) **Basic strategies:** To prove $P \equiv Q$, transform P to Q, transform Q to P, or transform $P \equiv Q$ to a previous theorem.

(3.26) **Heuristic:** Identify applicable theorems by matching the structure of subexpressions with the theorems.

(3.27) **Principle:** Structure proofs to avoid repeating the same subexpression on many lines.

(3.28) **Heuristic Unfold-fold:** To prove a theorem concerning an operator, remove it using its definition, manipulate, and reintroduce it using its definition.

(3.38) **Heuristic:** To prove $P \equiv Q$, transform the expression with the most structure (either P or Q) into the other.

(3.39) **Principle:** Structure proofs to minimize the number of rabbits.

(3.61) **Principle:** Lemmas can provide structure, bring to light interesting facts, and ultimately shorten proofs.

(3.60) **Heuristic:** Exploit the ability to parse theorems like the Golden rule in many different ways.

(4.10) **Deduction:** To prove $P \Rightarrow Q$, assume P and prove Q, with the variables and metavariables of P treated as constants.

(4.12) **Case analysis:** To prove S, prove $P \vee Q \vee R$, $P \Rightarrow S$, $Q \Rightarrow S$, and $R \Rightarrow S$.

(4.13) **Partial evaluation:** To prove $E[V := P]$, prove $E[V := \mathit{true}]$ and $E[V := \mathit{false}]$.

(4.14) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.19) **Contradiction:** To prove P, prove $\neg P \Rightarrow \mathit{false}$ (perhaps by assuming $\neg P$ and proving false).

(4.22) **Contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.

(5.2) **Heuristic:** To show that an English argument is sound, formalize it and prove that its formalization is a theorem.

(5.3) **Heuristic:** To disprove an expression (prove that it is not a theorem), find a counterexample: a state in which the expression is false.

General Laws of Quantification

For symmetric and associative binary operator $\star$ with identity $\mathbf{u}$.

(8.18) **Leibniz:** $$\frac{R \;\Rightarrow\; P = Q}{(\star x \mid R : E[V := P]) = (\star x \mid R : E[V := Q])}$$

(8.19) **Caveat.** The axioms and theorems for all quantifications hold only in cases in which all the individual quantifications are defined or specified.

Theorems

(8.20) **Axiom, Empty range:** $(\star x \mid \mathit{false} : P) \;=\; \mathbf{u}$

(8.21) **Axiom, Identity accumulation:** $(\star x \mid R : \mathbf{u}) \;=\; \mathbf{u}$

(8.22) **Axiom, One-point rule:** Provided $\neg occurs(\text{`}x\text{'}, \text{`}E\text{'})$,
$(\star x \mid x = E : P) \;=\; P^x_E$

(8.23) **Axiom, Distributivity:**
$(\star x \mid R : P) \star (\star x \mid R : Q) \;=\; (\star x \mid R : P \star Q)$

(8.24) **Axiom, Range split:**
$(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) \;=\; (\star x \mid R : P) \star (\star x \mid S : P)$

(8.25) **Axiom, Interchange of dummies:** Provided $\neg occurs(\text{'}y\text{'}, \text{'}R\text{'})$ and $\neg occurs(\text{'}x\text{'}, \text{'}Q\text{'})$,
$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$

(8.26) **Axiom, Nesting:** Provided $\neg occurs(\text{'}y\text{'}, \text{'}R\text{'})$,
$(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$

(8.27) **Range split:** Provided $R \wedge S \equiv \mathit{false}$
$(\star x \mid R \vee S : P) \;=\; (\star x \mid R : P) \star (\star x \mid S : P)$

(8.28) **Range split for idempotent $\star$:**
$(\star x \mid R \vee S : P) \;=\; (\star x \mid R : P) \star (\star x \mid S : P)$

(8.29) **Dummy reordering:** $(\star x, y \mid R : P) = (\star y, x \mid R : P)$

(8.30) **Change of dummy:** Provided $\neg occurs(\text{'}y\text{'}, \text{'}R, P\text{'})$, and f has an inverse,
$(\star x \mid R : P) = (\star y \mid R^x_{f.y} : P^x_{f.y})$

(8.31) **Dummy renaming:** Provided $\neg occurs(\text{'}y\text{'}, \text{'}R, P\text{'})$,
$(\star x \mid R : P) = (\star y \mid R^x_y : P^x_y)$

(8.32) **Split off term (example):**
$(\star i \mid 0 \le i < n+1 : P) \;=\; (\star i \mid 0 \le i < n : P) \star P^i_n$

Theorems of the Predicate Calculus

Universal quantification

(9.3) **Axiom, Distributivity of $\vee$ over $\forall$:** Provided $\neg occurs(\text{'}x\text{'}, \text{'}P\text{'})$,
$P \vee (\forall x \mid R : Q) \;\equiv\; (\forall x \mid R : P \vee Q)$

(9.4) **Trading:**
(a) $(\forall x \mid R : P) \;\equiv\; (\forall x \mid : R \Rightarrow P)$
(b) $(\forall x \mid R : P) \;\equiv\; (\forall x \mid : R \vee P \;\equiv\; P)$
(c) $(\forall x \mid R : P) \;\equiv\; (\forall x \mid : R \wedge P \;\equiv\; R)$
(d) $(\forall x \mid R : P) \;\equiv\; (\forall x \mid : \neg R \vee P)$

(9.5) **Trading:**
(a) $(\forall x \mid Q \wedge R : P) \;\equiv\; (\forall x \mid Q : R \Rightarrow P)$
(b) $(\forall x \mid Q \wedge R : P) \;\equiv\; (\forall x \mid Q : R \vee P \;\equiv\; P)$
(c) $(\forall x \mid Q \wedge R : P) \;\equiv\; (\forall x \mid Q : R \wedge P \;\equiv\; R)$
(d) $(\forall x \mid Q \wedge R : P) \;\equiv\; (\forall x \mid Q : \neg R \vee P)$

(9.6) **$\forall$-True body:** $(\forall x \mid R : \mathit{true}) \;\equiv\; \mathit{true}$

(9.7) **$\forall$-False body:** $(\forall x \mid : \mathit{false}) \;\equiv\; \mathit{false}$

(9.8) **$\forall$-Split-off-term:** $(\forall x \mid : P) \;\equiv\; (\forall x \mid : P) \wedge P^x_E$

(9.9) **Quantified freshy:** Provided $\neg occurs(\text{'}x\text{'}, \text{'}P\text{'})$,
$P \;\Rightarrow\; (\forall x \mid R : P)$

(9.10) Provided $\neg occurs(\text{'}x\text{'}, \text{'}P\text{'})$,
$P \wedge (\forall x \mid R : Q) \;\Rightarrow\; (\forall x \mid R : P \wedge Q)$

(9.11) **Distributivity of $\wedge$ over $\forall$:** Provided $\neg occurs(\text{'}x\text{'}, \text{'}P\text{'})$,
$\neg(\forall x \mid : \neg R) \;\Rightarrow\; ((\forall x \mid R : P \wedge Q) \;\equiv\; P \wedge (\forall x \mid R : Q))$

(9.12) $(\forall x \mid R : P \;\equiv\; Q) \;\Rightarrow\; ((\forall x \mid R : P) \;\equiv\; (\forall x \mid R : Q))$

(9.13) Provided $\neg occurs(`x', `Q')$,
$Q \Rightarrow (\forall x \mid R : P) \equiv (\forall x \mid R : Q \Rightarrow P)$

(9.14) **Monotonic ∀-body:**
$(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\forall x \mid R : P) \Rightarrow (\forall x \mid R : Q))$

(9.15) **Antimonotonic ∀-range:**
$(\forall x \mid \neg R : P \Rightarrow Q) \Rightarrow ((\forall x \mid P : R) \Leftarrow (\forall x \mid Q : R))$

(9.17) **Instantiation:** $(\forall x \mid : P) \Rightarrow P^x_E$

(9.18) Provided $\neg occurs(`x', `P')$, $(\forall x \mid : P) \equiv P$

(9.21) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

Existential quantification

(9.22) **Axiom, Generalized De Morgan:**
$(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

(9.23) **Generalized De Morgan:** (a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$
(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$
(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$

(9.25) **Distributivity of ∧ over ∃:** Provided $\neg occurs(`x', `P')$,
$P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$

(9.26) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.27) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$

(9.28) **∃-False body:** $(\exists x \mid R : \mathit{false}) \equiv \mathit{false}$

(9.29) **∃-True body:** $(\exists x \mid : \mathit{true}) \equiv \mathit{true}$

(9.30) **∃-Split-off-term:** $(\exists x \mid : P) \equiv (\exists x \mid : P) \vee P^x_E$

(9.31) **Quantified freshy:** Provided $\neg occurs(`x', `P')$,
$(\exists x \mid R : P) \Rightarrow P$

(9.32) Provided $\neg occurs(`x', `P')$,
$(\exists x \mid R : P \vee Q) \Rightarrow P \vee (\exists x \mid R : Q)$

(9.33) **Distributivity of ∨ over ∃:** Provided $\neg occurs(`x', `P')$,
$(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$

(9.34) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\exists x \mid R : P) \equiv (\exists x \mid R : Q))$

(9.35) Provided $\neg occurs(`x', `Q')$,
$(\exists x \mid R : P) \Rightarrow Q \equiv (\forall x \mid R : P \Rightarrow Q)$

(9.36) **Monotonic ∃-body:**
$(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\exists x \mid R : P) \Rightarrow (\exists x \mid R : Q))$

(9.37) **Monotonic ∃-range:**
$(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\exists x \mid P : R) \Rightarrow (\exists x \mid Q : R))$

(9.38) **∃-Introduction:** $P^x_E \Rightarrow (\exists x \mid : P)$

(9.39) Provided $\neg occurs(`x', `P')$, $(\exists x \mid : P) \equiv P$

(9.40) **Interchange of quantifications:**
Provided $\neg occurs(`y', `R')$ and $\neg occurs(`x', `Q')$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

(9.41) **Metatheorem Witness:** Provided $\neg occurs(`\hat{x}', `P, Q, R')$,
$(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P)^x_{\hat{x}} \Rightarrow Q$ is a theorem.

Set Theory

Set comprehension and enumeration

(10.2) **Set enumeration:**
$\{E_0, \ldots, E_{n-1}\} = \{x \mid x = E_0 \vee \cdots \vee x = E_{n-1} : x\}$

(10.3) **Axiom, Universe:** $\mathbf{U}{:}set(t) = \{x{:}t \mid true : x\}$

(10.4) **Axiom, Set membership:** Provided $\neg occurs(`x', `F')$,
$F \in \{x \mid R : E\} \equiv (\exists x \mid R : F = E)$

(10.5) **Axiom, Extensionality:** $S = T \equiv (\forall x \mid : x \in S \equiv x \in T)$

(10.6) $S = \{x \mid x \in S : x\}$

(10.7) **Empty range:** $\{x \mid false : P\} = \{\}$

(10.8) **Empty set:** $(v \in \{x \mid false : P\}) = (v \in \{\}) = false$

(10.9) **Singleton membership :** $e \in \{E\} \equiv e = E$

(10.10) **One-point rule:** Provided $\neg occurs(`x', `E')$,
$\{x \mid x = E : P\} = \{P[x{:=}E]\}$

(10.11) **Dummy reordering:** $\{x, y \mid R : P\} = \{y, x \mid R : P\}$

(10.12) **Change of dummy:** Provided $\neg occurs(`y', `R, P')$
and f has an inverse,
$\{x \mid R : P\} = \{y \mid R[x{:=}f.y] : P[x{:=}f.y]\}$

(10.13) **Dummy renaming:** Provided $\neg occurs(`y', `R, P')$,
$\{x \mid R : P\} = \{y \mid R[x{:=}y] : P[x{:=}y]\}$

(10.14) Provided $\neg occurs(`y', `R, E')$, $\{x \mid R : E\} = \{y \mid (\exists x \mid R : y = E)\}$

(10.15) $x \in \{x \mid R\} \equiv R$

(10.16) **Principle of comprehension.** To each predicate R corresponds the set $\{x{:}t \mid R\}$, which contains the objects in t that satisfy characteristic predicate R of the set.

(10.17) $\{x \mid Q\} = \{x \mid R\} \equiv (\forall x \mid : Q \equiv R)$

(10.18) **Metatheorem:** $\{x \mid Q\} = \{x \mid R\}$ is valid iff $Q \equiv R$ is valid.

Operations on sets

(10.20) **Axiom, Size:** $\#S = (\Sigma x \mid x \in S : 1)$

(10.21) **Axiom, Subset:** $S \subseteq T \equiv (\forall x \mid x \in S : x \in T)$

(10.22) **Axiom, Proper subset:** $S \subset T \equiv S \subseteq T \wedge S \neq T$

(10.23) **Axiom, Superset:** $T \supseteq S \equiv S \subseteq T$

(10.24) **Axiom, Proper superset:** $T \supset S \equiv S \subset T$

(10.25) **Axiom, Complement:** $x \in \sim S \equiv x \notin S$ (for x in $\mathbf{U}$)

(10.26) **Axiom, Union:** $v \in S \cup T \equiv v \in S \vee v \in T$

(10.27) **Axiom, Intersection:** $v \in S \cap T \ \equiv\ v \in S \land v \in T$

(10.28) **Axiom, Difference:** $v \in S - T \ \equiv\ v \in S \land v \notin T$

(10.29) **Axiom, Power set:** $v \in \mathcal{P}S \ \equiv\ v \subseteq S$

(10.31) **Metatheorem:** For any set expressions E_s and F_s,
(a) $E_s = F_s$ is valid iff $E_p \equiv F_p$ is valid,
(b) $E_s \subseteq F_s$ is valid iff $E_p \Rightarrow F_p$ is valid,
(c) $E_s = \mathbf{U}$ is valid iff E_p is valid.

PROPERTIES OF COMPLEMENT

(10.32) $\sim\{x{:}t \mid P\} \ =\ \{x{:}t \mid \neg P\}$

(10.33) $\sim\sim S = S$

(10.34) $\sim\mathbf{U} = \{\}$

(10.35) $\sim\{\} = \mathbf{U}$

PROPERTIES OF UNION AND INTERSECTION

(10.36)–(10.40): $\cup$ is symmetric, associative, idempotent, has zero $\mathbf{U}$, and has identity $\{\}$.

(10.41) **Weakening:** $S \subseteq S \cup T$

(10.42) **Excluded middle:** $S \cup \sim S = \mathbf{U}$

(10.43)–(10.47): $\cap$ is symmetric, associative, idempotent, has zero $\{\}$, and has identity $\mathbf{U}$.

(10.48) **Strengthening:** $S \cap T \subseteq S$

(10.49) **Contradiction:** $S \cap \sim S = \{\}$

(10.50) **Distributivity of $\cup$ over $\cap$:** $S \cup (T \cap V) \ =\ (S \cup T) \cap (S \cup V)$

(10.51) **Distributivity of $\cap$ over $\cup$:** $S \cap (T \cup V) \ =\ (S \cap T) \cup (S \cap V)$

(10.52) **De Morgan:** (a) $\sim(S \cup T) \ =\ \sim S \cap \sim T$
(b) $\sim(S \cap T) \ =\ \sim S \cup \sim T$

(10.53) $S \subseteq T \land W \subseteq V \ \Rightarrow\ (S \cup W) \subseteq (T \cup V)$

(10.54) $S \subseteq T \land W \subseteq V \ \Rightarrow\ (S \cap W) \subseteq (T \cap V)$

(10.55) $S \subseteq T \ \equiv\ S \cup T = T$

(10.56) $S \subseteq T \ \equiv\ S \cap T = S$

(10.57) $S \cup T = \mathbf{U} \ \equiv\ (\forall x \mid x \in \mathbf{U} : x \notin S \ \Rightarrow\ x \in T)$

(10.58) $S \cap T = \{\} \ \equiv\ (\forall x \mid : x \in S \ \Rightarrow\ x \notin T)$

(10.59) **Range split:** $\{x \mid P \lor Q : E\} = \{x \mid P : E\} \cup \{x \mid Q : E\}$

PROPERTIES OF SET DIFFERENCE

(10.60) $S - T = S \cap \sim T$

(10.61) $S - T \subseteq S$

(10.62) $S - \{\} = S$

(10.63) $S - \mathbf{U} = \emptyset$

(10.64) $\{\} - S = \{\}$

(10.65) $\mathbf{U} - S = \sim S$

(10.66) $S \cap (T - S) = \{\}$

(10.67) $S \cup (T - S) = S \cup T$

(10.68) $S - (T \cup V) = (S - T) \cap (S - V)$

(10.69) $S - (T \cap V) = (S - T) \cup (S - V)$

(10.70) $(S \cup T) - V = (S - V) \cup (T - V)$

(10.71) $(S \cap T) - V = (S - V) \cap (T - V)$

(10.72) $\sim(S - T) = \sim S \cup T$

Properties of subset

(10.73) $(\forall x \mid : P \Rightarrow Q) \equiv \{x \mid P\} \subseteq \{x \mid Q\}$

(10.74) **Antisymmetry:** $S \subseteq T \wedge T \subseteq S \equiv S = T$

(10.75) **Reflexivity:** $S \subseteq S$

(10.76) **Transitivity:** $S \subseteq T \wedge T \subseteq V \Rightarrow S \subseteq V$

(10.77) $\{\} \subseteq S$

(10.78) $S \subset T \equiv S \subseteq T \wedge \neg(T \subseteq S)$

(10.79) $S \subset T \equiv S \subseteq T \wedge (\exists x \mid x \in T : x \notin S)$

(10.80) $S \subseteq T \equiv S \subset T \vee S = T$

(10.81) $S \not\subset S$

(10.82) $S \subset T \Rightarrow S \subseteq T$

(10.83) $S \subset T \Rightarrow T \not\subseteq S$

(10.84) $S \subseteq T \Rightarrow T \not\subset S$

(10.85) $S \subseteq T \wedge \neg(V \subseteq T) \Rightarrow \neg(V \subseteq S)$

(10.86) $T \subseteq U \wedge \neg(T \subseteq S) \Rightarrow \neg(U \subseteq S)$

(10.87) **Transitivity:** (a) $S \subseteq T \wedge T \subset V \Rightarrow S \subset V$
(b) $S \subset T \wedge T \subseteq V \Rightarrow S \subset V$
(c) $S \subset T \wedge T \subset V \Rightarrow S \subset V$

Properties of powerset

(10.88) $\mathcal{P}\{\} = \{\{\}\}$

(10.89) $S \in \mathcal{P}S$

(10.90) $\#(\mathcal{P}S) = 2^{\# S}$ (for finite set S)

(10.91) **Monotonic** $\mathcal{P}$: $S \subseteq T \Rightarrow \mathcal{P}S \subseteq \mathcal{P}T$

(10.95) **Axiom of Choice:** For t a type, there exists a function $f{:}\,set(t) \rightarrow t$ such that for any nonempty set S, $f.S \in S$.

Mathematical Induction

(11.3) **Mathematical Induction over $\mathbb{N}$:** For all P,
$P.0 \wedge (\forall n{:}\mathbb{N} \mid : (\forall i \mid 0 \le i \le n : P.i) \Rightarrow P(n+1)) \Rightarrow (\forall n{:}\mathbb{N} \mid : P.n)$

(11.7) **Heuristic:** In proving $(\forall i \mid 0 \le i \le n : P.i) \Rightarrow P(n+1)$ by assuming $(\forall i \mid 0 \le i \le n : P.i)$, manipulate or restate $P(n+1)$ in order to expose at least one of $P.0, \ldots, P.n$.

(11.16) **Mathematical Induction over $\langle U, \prec \rangle$:** For all P,
$(\forall x \mid : P.x) \equiv (\forall x \mid : (\forall y \mid y \prec x : P.y) \Rightarrow P.x)$

(11.18) **Definition of well-founded:** $\langle U, \prec \rangle$ is well-founded iff, for all S,
$S \ne \{\} \equiv (\exists x \mid : x \in S \wedge (\forall y \mid y \prec x : y \notin S))$

(11.19) **Theorem:** $\langle U, \prec \rangle$ is well founded iff it admits induction.

(11.21) **Axiom, Finite chain property:**
$(\forall x \mid : (\forall y \mid y \prec x : DCF.y) \Rightarrow DCF.x)$

(11.23) **Theorem:** $\langle U, \prec \rangle$ is well founded iff every decreasing chain is finite.