Abstract

Real–world problems are often not fully characterized by a single optimal solution, as they frequently involve multiple competing objectives; it is therefore important to identify the so-called Pareto frontier, which captures solution trade-offs. We propose a fully polynomial-time approximation scheme based on Dynamic Programming (DP) for computing a polynomially succinct curve that approximates the Pareto frontier to within an arbitrarily small $\epsilon > 0$ on tree-structured networks. Given a set of objectives, our approximation scheme runs in time polynomial in the size of the instance and $1/\epsilon$. We also propose a Mixed Integer Programming (MIP) scheme to approximate the Pareto frontier. The DP and MIP Pareto frontier approaches have complementary strengths and are surprisingly effective. We provide empirical results showing that our methods outperform other approaches in efficiency and accuracy. Our work is motivated by a problem in computational sustainability concerning the placement of hydropower dams throughout the Amazon basin. Our goal is to support decision-makers in evaluating impacted ecosystem services on the full scale of the Amazon basin. Our work is general and can be applied to approximate the Pareto frontier of a variety of multiobjective problems on tree-structured networks.

1 Introduction

In recent years there has been considerable interest in the study of multi-objective optimization problems (see e.g., (Ehrgott and Gandibleux 2000; Qian, Tang, and Zhou 2016; Terra-Neves, Lynce, and Manquinho 2017; Wieck et al. 2008)). Multi-objective optimization is critical in Computational Sustainability (Gomes 2009), as real-world sustainability problems often involve balancing environmental, economic, and societal objectives. In multi-objective optimization, solutions are evaluated with respect to several, often competing, criteria. We are interested in understanding the trade-offs between the objectives for different solutions: the so-called Pareto frontier.

Our work is motivated by a challenging real-world computational sustainability problem concerning the placement of hydropower dams in a river network. In recent years, there has been a significant proliferation of hydropower dams in the Amazon basin: around 500 dams are proposed, planned, or under construction. Hydropower dams affect many ecosystem services such as, energy production, navigation, biodiversity, sediment export, nutrient cycling, and freshwater fisheries. Therefore, it is impera-
itive to integrate ecosystem service trade-offs into decision-making models for the placement of hydropower dams in the Amazon Basin (Finer and Jenkins 2012; Kareiva 2012; Winemiller et al. 2016; Zarfl et al. 2015; Ziv et al. 2012).

The Amazon hydropower dam placement problem represents a multiobjective optimization problem naturally captured by a tree-structured network. We are interested in mapping the Pareto frontier, since a single optimal solution does not fully characterize the problem: optimizing one objective often sacrifices the values of other objectives. The Pareto frontier is the set of all Pareto-optimal solutions, where a solution is considered Pareto-optimal if its vector of objective values is not dominated by the corresponding vector of any other feasible solution. Often the Pareto frontier is of exponential size, making it challenging to compute. By introducing an approximation of the exact frontier, we can replace the full frontier by a polynomial number of approximate Pareto points. Our approximate frontier is such that each Pareto solution in the full frontier is within a guaranteed approximation ratio from a point in the approximate frontier. Our approximation therefore provides an effective polynomial representation of the full frontier. Existing approaches are primarily heuristic-based and provide no formal guarantees (e.g., (Wieck et al. 2008; Ehrgott and Gandibleux 2000)). We develop efficient algorithms for approximating the Pareto frontier for tree-structured networks with a guarantee on the approximation quality.

Our contributions: 1) We provide an exact algorithm based on dynamic programming (DP) to compute the Pareto frontier on tree-structured networks. 2) A DP-based fully polynomial-time approximation scheme (FPTAS) for approximating the Pareto frontier for tree-structured networks. 3) We formulate the hydropower dam placement problem as a multiobjective optimization problem on a tree-structured network and demonstrate how our FPTAS approach can be applied to approximate its Pareto frontier. 4) We also formulate the problem of optimizing the placement of hydropower dams as a mixed integer programming problem (MIP) and use it to approximate the Pareto frontier following the scheme proposed in (Papadimitriou and Yannakakis 2000). 5) We provide empirical results on different basins of the Amazon to show interesting trade-offs between the two approaches. In general, our FPTAS captures the Pareto frontier more accurately than the MIP approach and provides a substantially larger set of Pareto frontier solutions. Nevertheless, in some cases, given that the MIP considers a smaller set of Pareto solutions (one solution per cell of the so-called ϵ-hyper-rectangle), it can take less time to approximate the Pareto frontier, despite its worst case exponential complexity. 6) We also show that our exact DP approach can compute the full (exact) Pareto frontier for several Amazon sub-basins and several criteria, which is infeasible for the MIP approach.

Our DP and MIP Pareto frontier approximations have complementary strengths and are surprisingly effective, scaling up to real-world size problems: the DP approximates the Pareto frontier for the entire Amazon basin, when optimizing for energy, seismic risk, and connectivity (a proxy for e.g., fish migrations, transportation, sediment production), in around 1.7 hours, with a coverage of 19955 non-dominated solutions, with the guarantee that the solutions are within at most 10% of the true optimum (ε = 0.1); in around 3 minutes, the DP provides a coverage of 3521 non-dominated solutions (ε = 0.2); in around 1 minute, the DP provides a coverage of 1976 non-dominated solutions (ε = 0.25); the MIP approach can approximate the Pareto frontier faster (≈ 16 seconds), but with a smaller coverage of 257 non-dominated solutions (ε = 0.1).

Our approach will enable policy-makers to make better informed decisions concerning the trade-offs of socioeconomic and environmental impacts of hydropower dam placements. We are developing analytical and visualization tools for an interactive exploration of the frontier for policy makers. The tools will allow policy makers to explore the trade-offs among the various solutions.

In the next sections we provide a formal characterization of the concept of Pareto frontier, describe our proposed FPTAS and MIP approaches for computing the Pareto frontier on tree-structured networks, and provide empirical results for different basins of the Amazon, demonstrating the trade-offs between the approaches.

2 Problem Statement

Pareto Frontier: We consider the trade-offs in optimizing multiple objective functions \((z^1, z^2, \ldots, z^m)\), whose values depend on a common solution (referred to as a policy \(π\)). In our application domain, a policy represents the selection of a particular set of dams to be built. Given such a policy (set of dams), we obtain values for our objective functions such as energy produced, seismic risk, etc. Without loss of generality, we only consider the problem of maximizing objective functions. Minimizing objective functions can be treated similarly. A Pareto optimal policy is one that is not dominated by any other policy. Given two policies \(π\) and \(π'\), we say that \(π\) dominates \(π'\) if the following two conditions hold: (1) for all \(i\), \(z^i(π) \geq z^i(π')\); (2) at least one strict inequality holds for some \(i\). The goal is to find all Pareto-optimal policies, which forms a set called the Pareto frontier.

Example: In our application domain, some objective functions that we consider are Energy (total energy generated by the dams), Connectivity (total river length from root without dams), and Seismic Risk (the overall seismic risk due to the constructed dams). These objective values are a direct function of the choice of dam placement and the structure of the underlying river network. We will define these objectives formally in later sections. For now, consider a very small example with only three possible dam placements P1, P2, and P3, with objective values \((10, 5, 35)\) (i.e., Energy = 10, Connectivity = 5, and Seismic Risk = 35), \((10, 3, 35)\), and \((11, 4, 34)\), respectively. The policy P1 dominates P2 because of the higher Connectivity. P1 and P3 are non-dominated policies and form the Pareto frontier. (Note that P1 does not dominate P3 because of the Energy value, and P3 does not dominate P1 because of the Connectivity and the Seismic Risk.) See figure 2 for an example of a visualization of the Pareto frontier. In the next paragraph we define the problem formally.
Definition of the Value Function:

A value function $z_i\left(\pi_i\right)$ is defined recursively on a tree network $T = \left(V, E\right)$ as an indicator function. The value function for a node $i$, i.e., $z_i\left(\pi_i\right) = 1$ if and only if the node $i$ is a root node without passing a selected dam site in the tree network $T$. All children of $i$, denoted as $\text{ch}(i)$, are leaf nodes if their children sets are empty.

The value function for a leaf node $i$ is defined as $z_i\left(\pi_i\right) = r_i$. Each leaf node $i$ has a unique parent node $\text{parent}(i)$.

The value function for a non-leaf node $i$ is defined recursively:

$$z_i\left(\pi_i\right) = r_i + \sum_{v \in \text{ch}(i)} \left(I(\text{uv} \in \pi) p_{uv} + I(\text{uv} \notin \pi) q_{uv}\right) z_v\left(\pi_v\right).$$

(1)

Here, $I(\cdot)$ is an indicator function. The value function for the entire tree network $T$ is the value function on the root node $s$, i.e., $z(T) = z_s\left(\pi_s\right)$.

The value function defined in this way is general and captures many interesting families of functions. For example, the value function of a finite binary-state Markov Decision Process can be encoded in this way by unfolding the decision process into a chain, which is in a special tree form. The value function can also be equivalently written in a more concise way. Let $uv$ denote the path from $u$ to $v$. Define the probability that $u$ is connected to $v$ as

$$p_{u \rightarrow v}(\pi) = \prod_{e \in u \rightarrow v} \left(I(e \in \pi)p_e + I(e \notin \pi)q_e\right),$$

that is, the product of all passage probabilities of edges on the path from $u$ to $v$ depending on their states. Then,

$$z(T)(\pi) = \sum_{v \in V} p_{s \rightarrow v}(\pi)r_v.$$  

(2)

Given $m$ objective functions $(z_1, z_2, \ldots, z_m)$, each defined as value functions on a tree network $T$, our multi-objective optimization problem on a tree structured network is to find the Pareto frontier consisting of all non-dominated policies.

The Pareto multi-objective optimization problem is NP-hard even though it is defined on a tree. To prove the NP-completeness, consider the following decision problem: Pareto_Dec$(t_1, \ldots, t_m)$: given $t_1, \ldots, t_m$, is there one policy $\pi$ such that $z(\pi) \geq t_i$ for all $i$?

Theorem 1. Pareto_Dec is NP-complete even for two objectives.

Proof: See the full version of the paper (Wu et al. 2018).

Application to the Hydropower Dam Placement Problem: We refer to the problem of determining the optimal placement of hydropower dams as the hydropower dam placement problem. As discussed earlier, we seek to balance multiple social, economic, and ecological metrics. The hydropower dam placement problem fits naturally into the general framework of the multi-objective optimization problem defined in the previous section.

To encode this problem, we first transform the river network and potential dam locations into a directed tree. A node in the directed tree corresponds to a contiguous region of the river network, that is, all stream segments in a region that are connected without being blocked by potential dam sites. An edge in the directed tree corresponds to a potential dam site. Figure 3 provides an example of this transformation.

Our policy $\pi$ is a subset of potential dam sites that we decide to build. Since each potential dam site is represented by an edge, the policy $\pi$, which is a subset of potential dam sites, corresponds to a subset of $E$, i.e., $\pi \subseteq E$. Building dams changes the passage probabilities for the edges on the tree network. If we build a dam, then the corresponding edge in the tree network is in its first state; otherwise it is in its second state. Using this framework, we can encode many important social and economic objectives, as follows:

Longitudinal connectivity: For a given policy $\pi$, the connectivity of a river network is measured by the total length of the stream segments that one can travel starting from the root without passing a selected dam site in $\pi$. Suppose we use the $i^{th}$ objective to formulate the connectivity. We set

\[ \text{ Connectivity } = \sum_{u \in \text{root}} p_{u \rightarrow v}(\pi) \]

where $v$ is the leaf node.

Seismic risk: For a given policy $\pi$, the seismic risk of a river network is measured by the sum of the seismic risk of each dam site in $\pi$. Suppose we use the $i^{th}$ objective to formulate the seismic risk. We set

\[ \text{ Seismic Risk } = \sum_{u \in \text{root}} I(e \in \pi)p_e + I(e \notin \pi)q_e \]

where $e$ is the edge.

Sediment: For a given policy $\pi$, the sediment transport of a river network is measured by the total amount of sediment transported through each dam site in $\pi$. Suppose we use the $i^{th}$ objective to formulate the sediment transport. We set

\[ \text{ Sediment } = \sum_{u \in \text{root}} p_{u \rightarrow v}(\pi) \]

where $v$ is the leaf node.

Energy: For a given policy $\pi$, the energy level of a river network is measured by the sum of the energy level of each dam site in $\pi$. Suppose we use the $i^{th}$ objective to formulate the energy level. We set

\[ \text{ Energy } = \sum_{u \in \text{root}} I(e \in \pi)p_e + I(e \notin \pi)q_e \]

where $e$ is the edge.
to be the total lengths of all stream segments in the region represented by node \( v \) in the directed tree. We set \( p^e_u = 0 \) and \( q^e_u = 1 \) for all \( e \in E \), that is, we either acquire all upstream segments (when we do not build the dam corresponding to edge \( e \) or lose all of them (when we build the dam). It is easy to show that Equation (1) encodes the connectivity metric.

**Energy:** The total hydropower that is produced by a set of selected dam sites is \( \sum_{e \in E} h_e \), where \( h_e \) is the hydropower for the dam represented by edge \( e \). Therefore, selecting a dam site on edge \( e \) adds energy \( h_e \) to the total energy. For any dam site \( (u, v) \) that can produce hydropower \( h_{uv} \), we set \( p^u_{uv} = q^v_{uv} = 1 \). We move the energy \( h_{uv} \) to its parent \( u \). In other words, the reward at node \( u \), \( r^u_e \) now becomes \( r^u_e(\pi) = \sum_{v \in ch(u)} I(uv \in \pi)h_{uv} \). Notice that \( r^u_e \) depends on whether each edge \( (u, v) \) is in policy \( \pi \). The total energy can be calculated by the same recursion as in (1).

**Sediments:** This objective captures the total amount of sediment carried downstream. Dams stop a certain fraction of sediments from moving downstream. The formulation of this objective is very similar to connectivity: \( r^u_e \) represents the amount of sediment that is produced by the region encoded by a node \( v \); \( q^u_v = 1 \) and \( p^u_v = e \) is the percentage of sediment that can go through the dam denoted by edge \( e \).

**Seismic Risk:** Dam locations are associated with three levels of seismic risk: level 1 (no risk) to 2 (medium risk) or 3 (high risk). Let \( h_e \) be the hydropower produced by dam \( e \), \( l_e \) be the risk level for its location, and \( \eta = 10 \) be a base risk score. We define the seismic hazard for dam \( e \) as zero if the dam is located in a level 1 region \( (l_e = 1) \); otherwise its risk is \( (h_e + C)l^e \), which can be encoded analogously to the energy objective. \( C \) is a constant. We set it to 1000.

Many other objectives can be encoded in a similar way.

### 3 Dynamic Programming and Rounding

We first present a dynamic programming algorithm which computes the Pareto frontier exactly for value functions defined on a tree-structured network. We prove that the algorithm can find all Pareto optimal policies.

In practice, the number of Pareto optimal policies may be exponential even for a fixed number of objective functions. Motivated by the ideas in (Wu, Sheldon, and Zilberstein 2014a; 2014b), we apply a rounding technique to the dynamic programming algorithm. (Wu et al. do not consider the problem of approximating the Pareto frontier.) We provide a fully polynomial-time approximation scheme (FP-TAS) that finds a policy set of polynomial size, which approximates the Pareto frontier within an arbitrary small \( \epsilon \) and runs in time polynomial in the size of the instance and \( 1/\epsilon \).

**Definition 1.** We say that a policy set \( S \) \( \epsilon \)-approximates the Pareto frontier, if and only if for any policy \( \pi \) in the Pareto frontier, there exists a policy \( \pi' \) in set \( S \), such that \( z^i(\pi') \geq (1 - \epsilon)z^i(\pi) \) for all \( i \).

**Definition 2.** We say an algorithm \( A \) is a fully polynomial-time approximation scheme (FP-TAS) for a multi-objective optimization problem, if \( A \) finds a policy set \( S \) that \( \epsilon \)-approximates the Pareto frontier, and \( A \) runs in time polynomial in the size of the instance and \( 1/\epsilon \).

As we will show below, our main contribution can be summarized by the following statement:

Suppose there are universal constants \( c \) and \( C \) such that \( c \leq r^u_e \leq C \) for all \( u \in V \) and all \( i \). Our algorithm \( \text{Pareto}_{\epsilon} \), described below, is a FP-TAS for the multiple objective optimization problems defined on a tree structured network. The size of policy set found by the algorithm is bounded by \( O \left( \left( \frac{n}{\epsilon} \right)^m \right) \), and the running time is bounded by \( O \left( \left( \frac{n}{\epsilon} \right)^{2m} \right) \), where \( n \) is the number of nodes in \( T \) and \( m \) is the number of objective functions.

We assume that the tree \( T \) is a binary tree. In section 5, we discuss how to transform a general directed rooted tree into an equivalent binary tree, which allows us to apply the algorithm discussed in this section with the additional benefit of increased early pruning.

#### 3.1 Exact Dynamic Programming Algorithm

Our proposed algorithm, \( \text{Pareto}_{\epsilon} \), is an exact dynamic programming algorithm, which computes all policies in the Pareto frontier. One may relate it to the bucket elimination algorithm in probabilistic inference (Dechter 1998). Nevertheless bucket elimination is applied to single objective problems; we note that having multiple objectives makes the problem structure very different from single objective problems. In fact, the complexity of variable elimination is bounded by the tree width, so tractable on trees. However, our problem is NP-hard even for two objectives and on a two-layer tree. In addition, we obtain an FP-TAS for our problem, which is not provided by the bucket elimination algorithm, even for single objective problems.

Let \( \Phi_u \) denote the set of all edges in the subtree rooted at \( u \) and \( \pi_u \) be a partial policy on \( u \) that only includes edges in \( \Phi_u \); in other words, \( \pi_u \subseteq \Phi_u \). A partial policy is Pareto optimal if and only if it is not dominated by any other partial
policies on the same node. Our dynamic programming algorithm is based on the following theorem, which states that all Pareto optimal partial policies on a node can be computed based on Pareto optimal partial policies from its children:

**Theorem 2.** Let $l$ and $r$ be the children of $u$. Any Pareto-optimal partial policy at $u$ can be constructed by combining one Pareto-optimal partial policy from node $l$, one from node $r$ and four different joint states of edge $(u, l)$ and $(u, r)$.

**Proof.** Prove by contradiction. See the full version of the paper (Wu et al. 2018).

Motivated by Theorem 2, we design our exact dynamic programming algorithm, Pareto$_T$, to recursively compute the Pareto optimal partial policies from leaf nodes to the root. Instead of keeping all partial policies, we only keep the Pareto-optimal partial policies at each node. The recursion of computing the Pareto optimal partial policy for node $u$ is as follows. Suppose $l$ and $r$ are the two children of node $u$. We first compute the Pareto optimal partial policies for $l$ and $r$. Then we enumerate each pair of partial policies from node $l$ and $r$ and consider four different combinations of whether to include edge $(u, l)$ and $(u, r)$. For each combination, we compute the objective values based on Equation 1, and add them to the policy set $P$. Finally, we remove all dominated policies from $P$. Based on Theorem 2, the remaining policies are Pareto optimal for node $u$. Note that it is straightforward to adapt this algorithm to include the case when the node $u$ only has one child.

### 3.2 Approximation with Rounding

In practice, the optimal Pareto frontier may have an exponential number of points. In this case, we propose Pareto$_{approx_T}$, based on a rounding technique that gives a FPTAS to the multi-objective optimization problem.

We first explain how rounding works for longitudinal connectivity and sediments. The rounding for energy and seismic risk, which will be detailed later, is very similar to the rounding scheme presented here, except for an additional pre-rounding step to take into account the fact that the node reward varies according to policy.

We introduce a hyper-parameter $K_u^i$ for each node and each objective. Define a rounded objective value $\hat{z}_u^i(\pi)$, which is calculated by the following recursion,

$$\hat{z}_u^i(\pi) = r_u^i + \left[ \frac{\sum_{v \in Ch(u)} (I(uv \in \pi) p_{uv}^i + I(uv \notin \pi) q_{uv}^i) \hat{z}_v^i(\pi)}{K_u^i} \right] K_u^i.$$  

(3)

In Equation 3, the fraction less than $K_u^i$ is removed so that the number of different rounded objective values is reduced. This idea is similar to rounding different fractional numbers to the same integer value.

The rounded dynamic programming algorithm Pareto$_{approx_T}$, is similar to the DP algorithm, Pareto$_T$, except that we use Equation 3 to calculate objective values.

**Proposition 1.** $z_i^i(\pi) \geq \hat{z}_u^i(\pi)$ for any hyperparameter $K_u^i$ and any policy $\pi$.

This proposition is a direct consequence of taking the floor operator in Equation 3.

**Proposition 2.** If we set $K_u^i = cr_u^i$, for any policy $\pi$ we have

$$z_i^i(\pi) - \hat{z}_u^i(\pi) \leq \epsilon z_i^i(\pi).$$

To see why this proposition is true, at each node, we at most lose a value of $K_u^i$. Based on Equation. 2, the total value we lost due to the floor operation in Equation. 3 is

$$z_i^i(\pi) - \hat{z}_u^i(\pi) \leq \sum_{v \in V} p_{s-v}(\pi) K_u^i = \epsilon z_i^i(\pi).$$  

(4)

**Proposition 3.** The approximate algorithm computes all Pareto optimal points for the rounded objective.

This is because the approximate algorithm Pareto$_{approx_T}$ is the same as the exact algorithm Pareto$_T$, except for replacing the exact objective function with a rounded one. The correctness of Pareto$_T$ does not depend on particular objective function. Therefore, this proposition is true.

**Theorem 3.** Let $P(s)$ be the set of (partial) Pareto optimal policies for a node $s$. Let $\overline{P}(s)$ be the set of approximate (partial) Pareto optimal policies computed via Pareto$_{approx_T}$ with rounded objective function (Equation 3). We must have $\overline{P}(s) \epsilon$-approximates $P(s)$.

**Proof.** Suppose $\pi$ is in the exact Pareto frontier $P(s)$. If $\pi$ is in $\overline{P}(s)$, we are done. Otherwise, because of Proposition 3, there must be at least one $\pi' \in \overline{P}(s)$, such that $\pi'$ dominates $\pi$ in terms of the rounded objective; i.e.,

$$\hat{z}_u^i(\pi') \geq \hat{z}_u^i(\pi),$$  

(5)

for all $i$. Because of Proposition 1, we have

$$z_i^i(\pi') \geq \hat{z}_u^i(\pi').$$  

(6)

Because of Proposition 2, we have

$$\hat{z}_u^i(\pi') \geq (1 - \epsilon) z_i^i(\pi').$$  

(7)

Combining Equation (5) (6) and (7), we have

$$z_i^i(\pi') \geq (1 - \epsilon) z_i^i(\pi).$$  

(8)

This concludes the proof.

We are also able to show that Pareto$_{approx_T}$ runs in polynomial amount of time. To show this, we have to bound the number of partial policies in $\overline{P}(u)$ for each node $u$. In practice, we can assume that the objective functions are bounded.

**Assumption 1.** There are universal constants $c$ and $C$ such that $c \leq r_u^i \leq C$ for all $u \in V$ and all $i$.

**Theorem 4.** With Assumption 1, the number of different rounded tuples in $\overline{P}(u)$ is $O\left( \left( \frac{n_u}{\epsilon} \right)^m \right)$, where $n_u$ is the number of nodes in the subtree rooted at $u$. 

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Proof. The upper bound of the $i^{th}$ objective value of $u$ is $UB_i^u = \sum_{v \in T_u} r_i^v$, with $T_u$ denoting the subtree rooted at $u$. Due to Assumption 1, the number of different tuples of $u$ is
\[ \prod_{i=1}^m UB_i^u \leq \prod_{i=1}^m n_u C_i \leq e \cdot \left( \left( \frac{n_u}{\epsilon} \right)^m \right). \tag{9} \]

\[ \Box \]

Theorem 5. The runtime of the rounded dynamic programming algorithm is $O \left( \left( \frac{n}{\epsilon} \right)^{2m} \right)$. Proof. As we proved, the number of different rounded tuples $T_i(u)$ at node $u$ is $O \left( \left( \frac{n_u}{\epsilon} \right)^m \right)$. Let $T_u$ be the runtime to compute all these tuples at $u$. We have the recursion
\[ T_u = O(\sum_{i=1}^m T_i) + T_1 + T_r. \tag{10} \]
Solving the recursion, we have $T_u = O \left( \left( \frac{n_u}{\epsilon} \right)^{2m} \right)$ where $n$ is the number of nodes in the tree. \[ \Box \]

For energy and seismic risk: We apply a two-step rounding scheme for energy and seismic risk. Notice it can be applied to connectivity and sediments to obtain the same approximation guarantee. For presentation purposes, we use energy as an example. In the first step, we round upfront the energy value $h_{uv}$ to
\[ \left\lfloor \frac{h_{uv}}{K_1} \right\rfloor K_1, \]
in which $K_1 = z h_{min}$, where $h_{min}$ is the minimal energy value among all dams. This rounding is similar to the one used in the FPTAS to solve the knapsack problem (see page 67 of (Williamson and Shmoys 2011)\textsuperscript{1}). In fact, one can prove that we already get an FPTAS, if we only apply the first-step rounding. The proof is similar to that of the knapsack rounding.

The rounding in the first step is often too conservative. To obtain additional pruning, we apply a second rounding step in our DP approach when combining partial solutions from the children for the parent node. The rounding looks exactly like the one we use for connectivity and sediments in (3)\textsuperscript{2}, except that we do not round when $r_i^u(\pi) = 0$. The second rounding is more aggressive. As more partial solutions are rounded to the same value, the algorithm is able to prune more solutions, therefore scaling to larger instances.

To prove the approximation guarantee of the two-step rounding, we show that the first and the second rounding each incurs at most a $\epsilon/2$-approximation. Therefore, the entire algorithm has an $\epsilon$-approximation guarantee. Let $z^*_u(\pi)$ be the true objective function without any rounding. $z^*_u(\pi)$ is the objective value after applying the first rounding. $z^*_u(\pi)$ is the objective value after applying both the first and the second rounding. We are able to prove that
\[ z^*_u(\pi) - z^*_u(\pi) \leq \epsilon z^*_u(\pi), \tag{11} \]
\[ z^*_u(\pi) - z^*_u(\pi) \leq \epsilon z^*_u(\pi). \tag{12} \]
Adding these two inequalities together, we get Proposition 2, which the proof of the approximation guarantee mainly depends on. The proof to Equation 11 is similar to that for the knapsack algorithm. The proof to Equation 12 is similar to the proof of Proposition 2 with connectivity and sediments as objectives.

To prove that the DP algorithm runs in polynomial amount of time, we notice that during the execution of the algorithm, all intermediary energy values are in the form of $k K_1$, where $k$ is an integer and $K_1$ is the rounding factor of the first step. As a consequence, the maximal number of different energy values at node $u$ are bounded by
\[ \frac{n_u h_{max}}{K_1} = \frac{2n_u h_{max}}{\epsilon h_{min}} = O \left( \frac{n_u}{\epsilon} \right) \]
where $n_u$ is the number of nodes for the subtree rooted at $u$. $h_{max}$ is the maximal energy value.

Theorem 6. Pareto approx$_\epsilon$ is an FPTAS for the multi-objective optimization problem on a tree structured networks.

We provide a detailed proof of FPTAS with the two-step rounding scheme in the full version of the paper (Wu et al. 2018).

4 Mixed Integer Programming

We also formulate the problem of optimizing the placement of hydropower dams as a mixed integer programming problem (MIP) and use it to approximate the Pareto frontier following the scheme proposed in (Papadimitriou and Yannakakis 2000). While the resulting method has exponential worst case complexity because of the cost of solving the MIP, we find that in practice it is complementary to the DP approximation.

The scheme proposed by (Papadimitriou and Yannakakis 2000), which finds a policy set that $\epsilon$-approximates the Pareto frontier, can be summarized as follows: for a multi-objective optimization problem, we divide the space of objectives into small rectangular cells and we query whether there exists a solution in each cell. We form a set $S$ that includes one solution for each rectangular cell where there is at least one solution. It can be proved that if the boundaries of the rectangular cells are carefully chosen, the set of non-dominated solutions from $S$ form a $\epsilon$-approximate Pareto frontier.

Specifically, the rectangular cells are designed to satisfy the condition that, for each dimension, the upper bound is $(1 + \epsilon)$ from the lower bound (assuming the objectives are always positive values). For our problem, we are considering four objectives: $C$ (longitudinal connectivity), $S$ (Sediment), $H$ (Hydropower) and $L$ (Seismic Risk). The four objectives are always positive. Assume that $C_{max}$ and $C_{min}$ are

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\textsuperscript{1}One difference: the rounding depends on the minimal energy value, rather than the maximal value as in the knapsack rounding.

\textsuperscript{2}This is because the value of the most valuable object is naturally a lower bound to the knapsack problem, but it is not the case in our multi-objective optimization problem.
the minimum and maximum possible connectivity values for any dam building scheme. Let \( C \) be the set
\[
\{ C_{\min} (1+\epsilon)^k \mid k = 0, 1, 2, \ldots K, C_{\min} (1+\epsilon)^{K_e+1} \geq C_{\max} \}
\]
\( \mathcal{L}, \mathcal{H} \) and \( \mathcal{S} \) are defined similarly. Then, each rectangular cell is of the form
\[
[\hat{C}, (1+\epsilon)\hat{C}] \times [\hat{S}, (1+\epsilon)\hat{S}] \times [\hat{H}, (1+\epsilon)\hat{H}] \times [\hat{L}, (1+\epsilon)\hat{L}]
\]
where \( \hat{C} \) ranges in \( C \), \( \hat{S} \) ranges in \( S \), \( \hat{H} \) ranges in \( H \), and \( \hat{L} \) ranges in \( L \). The decision problem that the MIP solves is to decide whether there is at least one solution for each rectangular cell.

We construct a MIP formulation of the decision problem. We use the following notations:

- \( V \): the set of all nodes.
- \( s \): the root of the tree (also the mouth of the river network).
- \( e = (u,v) \): \( u \) is the node downstream of edge \( e \) and \( v \) is the node upstream of \( e \).
- \( c_{uv} \): the total connectivity of river segments at node \( v \).
- \( s_v \): the total sediment produced at node \( v \).
- \( h_e \): hydropower of dam \( e \).
- \( \eta_e \): seismic risk level of dam \( e \).
- \( p_e \): percentage of sediment trapped by dam \( e \) if the dam is built.
- \( \pi_e \): indicator variable of MIP. It evaluates to 1 iff \( e \in \pi \).
- \( n_u \): indicator variable of MIP. It evaluates to 1 iff node \( v \) can be reached from the river mouth without passing a dam.
- \( y_v \): continuous variable in MIP. The percentage of the sediment produced at the node \( v \) not trapped by dams.

Here we show the MIP encoding considering the following objectives:

**Longitudinal Connectivity** (\( C \)). To define connectivity we assign a binary variable \( n_{uv} \in \{0,1\} \) to each node \( v \), which represents whether a node can be reached from the river mouth without passing a built dam. Therefore, the longitudinal connectivity can be defined as

\[
C = \sum_{v \in V} n_{uv}c_{uv}.
\]

Suppose \( s \) is the river mouth. For each node \( v \neq s \), \( n_{uv} = 1 \) if and only if at least one downstream node \( u \) satisfies \( n_{uv} = 1 \) and the dam \( e = (u, v) \) is not built (\( \pi_e = 0 \)). This fact can be encoded using the following constraints:

\[
\begin{align*}
n_u & = 1; n_{uv} \geq n_v, \forall (u,v) \in E \\
n_v & \leq 1 - \pi_{u,v}, \forall (u,v) \in E \\
n_v & \geq n_u - \pi_{u,v}, \forall (u,v) \in E
\end{align*}
\]

**Sediment** (\( S \)). For each node \( v \), we define a continuous variable \( y_v \in [0,1] \) that represents the percentage of the sediment produced at the node that is not trapped by downstream dams. At river mouth \( s \), \( y_s = 1 \). Also, \( y_v = \)

\[
\begin{align*}
\text{Minimize: } d & \\
\text{subject to: } d & = 0 \quad (d \text{ is a dummy variable}) \\
\hat{C} & \leq C \leq (1+\epsilon)\hat{C}; \hat{S} \leq S \leq (1+\epsilon)\hat{S} \\
\hat{H} & \leq H \leq (1+\epsilon)\hat{H}; \hat{L} \leq L \leq (1+\epsilon)\hat{L} \\
C & = \sum_{v \in V} n_{uv}c_{uv}; L = \sum_{e \in E} h_e\eta_e \\
H & = \sum_{e \in E} \pi_e h_e; S = \sum_{v \in V} y_v s_v
\end{align*}
\]

\[
\begin{align*}
y_v & \leq y_u \quad \text{if } \pi(i) = 0, \text{ This condition can be linearized using the big M method. Since } y_u \in [0,1] \text{ for all } v \in V, \text{ we can replace the condition with} \\
(1-p_e)y_u & \text{ otherwise.}
\end{align*}
\]

The energy output is encoded as \( H = \sum_{e \in E} \pi_e h_e \).

**Seismic Risk** (\( L \)). \( L \) can be encoded as \( L = \sum_{e \in E,L>0} (h_e + F)\eta_e \). (\( F \) is a constant. See earlier description of seismic risk.)

Summarizing the constraints above, we obtain a compact MIP encoding of the problem shown in Figure 4. Because the MIP solves a decision problem, we minimize a dummy variable \( d \). We run the MIP to see if there is a solution in every rectangular cell and we remove dominated solutions.

**5 Experimental Results**

**Data:** In order to test our methodological approach at different spatial scales, we developed three sub-sets of data that correspond to different geographical areas within the Amazon region: the western Amazon basin, the Marañón basin, and the entire Amazon basin (Finer and Jenkins 2012; Shedlock et al. 2000; Venticinque et al. 2016; Winemiller et
Table 1: The equivalent DP binary tree (Western Amazon) leads to considerable speed-ups: it considers many fewer policies than the original tree due to early pruning.

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Table 2: Examples of runtimes and number of solutions: Maraño (M; 109 nodes), Western Amazon (WA; 219 nodes) and Amazon (A; 468 nodes) showing the trade-offs between the different methods. ($S_d$ - sediment; $S_s$ seismic)

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Figure 5: Approximation quality: The DP approximations (for all considered $\epsilon$) nearly overlap the exact Pareto curve (blue). The MIP approximations lie farther from the exact Pareto-Frontier than the corresponding DP approximations do. The DP approximations also better capture the shape and coverage of the exact Pareto frontier.

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**Figure 5:** Approximation quality: The DP approximations (for all considered $\epsilon$) nearly overlap the exact Pareto curve (blue). The MIP approximations lie farther from the exact Pareto-Frontier than the corresponding DP approximations do. The DP approximations also better capture the shape and coverage of the exact Pareto frontier.
of Pareto optimal partial policies remains of reasonable size throughout the computation. See table 1.

**Exact Pareto frontier:** The DP approach can compute the exact Pareto Frontier (time permitting), which is infeasible for the MIP approach. We were surprised to see that the DP approach can compute the complete exact Pareto frontier, for connectivity and energy, for the entire Amazon basin, which contains 38459 solutions, in around 5 hours (17170 secs). (See figure 5.)

**Quality of the Pareto Frontier approximation:** We observe that, for the same $\epsilon > 0$, the accuracy of the DP approximation is in general better than the MIP approximation, in practice. This is illustrated in figure 5.

**Approximation Coverage of the Pareto Frontier:** The DP approach has a substantially better coverage of the exact Pareto frontier, compared to the MIP approximation, which is reflected in the larger number of solutions that it produces. See figures 5 and 6.

Evolutionary algorithms have been widely used to approximate (without approximation guarantees) Pareto frontiers (see e.g., Neumann 2007; Qian, Yu, and Zhou 2013; 2015; Qian, Tang, and Zhou 2016; Wiecek et al. 2008; Martens and Izzo 2013)). Here we include a comparison to the Non-dominated Sorting Genetic Algorithm (NSGA-II). From figure 5, we see that NSGA-II is comparable in accuracy to the DP approximations with $\epsilon = 0.1$ but takes longer to compute (and without approximation guarantees).

**Scalability:** The complementary strengths of our approximation paradigms are captured in Figure 7 and Table2, which includes a small sample of the runtimes and number of solutions for the DP and MIP approximations, for the three different networks that we considered: Maranon; Western Amazon; and entire Amazon basin. The MIP runtime is largely determined by the number of cells in the hypercube that need to be considered for feasibility, which is determined by $\epsilon$, the size of network, and number of criteria. Somewhat surprisingly, the size of the network does not appear to affect significantly the MIP feasibility checks. This phenomenon may be explained by the fact that most feasibility checks are either in under or over constrained areas, which are relatively easy to solve. Only a few cells are critically constrained. Our DP approximation, provides a good coverage of the solutions on the Pareto frontier. In contrast to the MIP, the DP approximation is largely dependent on the number of true Pareto optimal solutions, which generally scales with the size of the network and of course the number of criteria. Interestingly, the DP scales better than the MIP for the sediment objective, while the seismic objective is more challenging for DP than for MIP.

**Visualization of the Pareto Frontier:** For policy makers to make informed decisions based on the approximate Pareto frontier, we are developing a variety of analytical and visualization tools that allow for an interactive exploration of the frontier. The tools allow policy makers to explore the trade-offs among the various solutions. See figure 2.

### 6 Conclusions

We introduced a DP-based fully polynomial-time approximation scheme for computing a polynomially succinct curve that approximates the Pareto frontier to within an arbitrary small $\epsilon > 0$ on tree-structured networks. Given a set of objectives, our fully polynomial approximation scheme runs in time polynomial in the size of the instance and $1/\epsilon$. We also introduced a MIP-based scheme to approximate the Pareto frontier. Our work is motivated by a problem in computational sustainability concerning the placement of hydropower dams throughout the Amazon basin. The DP and MIP Pareto frontier approximations have complementary strengths and are surprisingly effective, scaling...
up to the entire Amazon basin (500 potential dam sites). For example, our DP can generate an approximate Pareto frontier that is within 10% of optimal and contains 19955 non-dominated solutions in roughly 1.7 hours, when optimizing for energy, connectivity, and seismic risk. A within 25% from optimal frontier can be generated in around 1 minutes and contains 1976 solutions. The MIP approach is even faster (16 seconds, within 10% from optimal) but provides less coverage of the frontier in terms of number of solutions (257 non-dominated solutions). Our overall goal is to support decision-makers in evaluating impacted ecosystem services on the full scale of the Amazon basin, which is unreachable for other methods. Moreover, our work is general and can be applied to approximate the Pareto frontier of a variety of multiobjective problems, e.g., those concerning finite Markov Decision Processes.

7 Acknowledgements

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References


