Decomposing Properties into Safety and Liveness using Predicate Logic†

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Decomposing Properties into Safety and Liveness

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ABSTRACT

A new proof is given that every property can be expressed as a conjunction of safety and liveness properties. The proof is in terms of first-order predicate logic.

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1. Introduction

Two classes of properties are of particular interest when considering programs: safety properties and liveness properties. Informally, a safety property stipulates that "bad things" do not happen during execution of a program and a liveness property stipulates that "good things" do happen (eventually) [2]. Distinguishing between safety and liveness properties is useful because knowing whether a property is safety or liveness helps when deciding how to prove that the property holds for a program.

In [1], formal definitions of safety and liveness are given and it is proved that every property can be expressed as the conjunction of a safety property and a liveness property. The formal definitions of safety and liveness are given in terms of first-order predicate logic, but the proof that every property can be decomposed into safety and liveness is not—it uses topology. The purpose of this paper is to give a proof of this theorem using only first-order predicate logic.

2. Specifying Properties

A program state is a mapping from variables to values. An execution of a concurrent program can be viewed as an infinite sequence of program states

$$\sigma = s_0 s_1 ..., $$

which we call a history. In a history, $$s_0$$ is an initial state of the program and each subsequent state results from executing a single atomic action in the preceding state. (For a terminating execution, an infinite sequence is obtained by repeating the final state.) A property is a set of such sequences.

One way to specify a property is by using first-order predicate logic. For a state $$s$$, define $$s.v$$ to be the value of variable $$v$$ in that state. A formula of first-order predicate logic where $$s$$ is the only free variable defines a set of states. For example,

$$(\forall i: 1 \leq i \leq N: s.a[i] \leq s.a[i+1])$$

specifies the set of states in which the elements of array $$a[1:N]$$ are sorted. Usually "$$s.$$" is implicit and therefore left out of such a formula, resulting in the more familiar use of first-order predicate logic as an assertion language.

A set of sequences of states—a property—can also be defined using first-order predicate logic. To facilitate such specifications, for any sequence $$\sigma = s_0 s_1 \ldots$$ define for $$0 \leq i$$:

$$\sigma[i] = s_i.$$  
$$\sigma[..] = s_0 s_1 \ldots s_{i-1}.$$ The empty sequence if $$i = 0.$$  
$$|\sigma| =$$ the length of $$\sigma$$ ($$\omega$$ if $$\sigma$$ is infinite).

A formula of first-order predicate logic in which $$\sigma$$ is the only free variable defines the set of sequences that satisfy the formula and therefore specifies a property. For example,

$$(\forall i: 0 \leq i: \sigma[i].v = 0)$$

specifies the property in which the value of $$v$$ remains 0 throughout execution.
We write $\alpha \models P$ if $\alpha \in S^\omega$ is in the property specified by $P$. Thus,

\begin{align*}
\alpha \models P &= \alpha \models P, \\
\alpha \not\models P &= \neg \alpha \models P.
\end{align*}

3. Safety and Liveness

According to [1], a property $P$ is a safety property provided

\begin{equation}
\text{Safety: } (\forall \alpha: \sigma \in S^\omega: \sigma \mid P \Rightarrow (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i\beta] \mid P))), \tag{3.1}
\end{equation}

where $S$ is the set of program states, $S^*$ the set of finite sequences of states, $S^\omega$ the set of infinite sequences of states, and juxtaposition is used to denote catenation of sequences. A property $P$ is a liveness property provided

\begin{equation}
\text{Liveness: } (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha \beta \mid P)). \tag{3.2}
\end{equation}

Given a property $P$, we are interested in defining properties $\text{Safe}(P)$ and $\text{Live}(P)$ such that

- $\text{Safe}(P)$ is a safety property,
- $\text{Live}(P)$ is a liveness property, and
- $P = \text{Safe}(P) \land \text{Live}(P)$.

Observe that if

\begin{align*}
\text{Safe}(P) &= P \lor M_P, \\
\text{Live}(P) &= P \lor \neg M_P
\end{align*}

then

\begin{align*}
\text{Safe}(P) \land \text{Live}(P) &= (P \lor M_P) \land (P \lor \neg M_P) \\
&= (P \land P) \lor (P \land \neg M_P) \lor (M_P \land P) \lor (M_P \land \neg M_P) \\
&= P
\end{align*}

Hence, we have only to look for an $M_P$ that makes $P \lor M_P$ (i.e. $\text{Safe}(P)$) a safety property and $P \lor \neg M_P$ (i.e. $\text{Live}(P)$) a liveness property.

It turns out that using

\begin{equation}
M_P: (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[i\beta] \mid P))
\end{equation}

suffices. First, we show formally that $\text{Safe}(P)$ satisfies definition (3.1) of safety. The proof that follows is a sequence of first-order predicate logic formulas with explanations interspersed (and delimited by « and ») of how each formula is derived from its predecessor.

Choose any $\sigma \in S^\omega$:

\begin{equation}
\sigma \mid \text{Safe}(P)
\end{equation}
»by definition of Safe(P)«
= \sigma\#(P \vee (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P)))
«by definition of \#»
= \neg (P \vee (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P)))^T
«by substitution»
= \neg (P \vee (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P)))
«by De Morgan's Laws»
= \neg P \land (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P))
«A \land B \Rightarrow B»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P))
«because (\forall x:: A) = (\forall x:: A \land (\forall y:: A_T))»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (\forall \gamma: \gamma \in S^\omega: (\sigma[i] \beta) \vdash \gamma P)))
«because true \land P = P and (\sigma[i] \beta)[i] = \sigma[i]»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (i = i) \land (\forall \gamma: \gamma \in S^\omega: (\sigma[i] \beta)[i] \gamma P)))
«by substitution»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (k = i) \land (\forall \gamma: \gamma \in S^\omega: (\sigma[i] \beta)[k] \gamma P)))
«by \exists i-Generalization»
⇒ (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (\exists k: k = i: (\forall \gamma: \gamma \in S^\omega: (\sigma[i] \beta)[k] \gamma P)))
«by Range Widening»
⇒ (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (\exists k: 0 \leq k: (\forall \gamma: \gamma \in S^\omega: (\sigma[i] \beta)[k] \gamma P)))
«by De Morgan's Law»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (\neg (\forall k: 0 \leq k: (\exists \gamma: \gamma \in S^\omega: (\sigma[i] \beta)[k] \gamma \vdash P)))
«by definition of \#»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P \land (\forall k: 0 \leq k: (\exists \gamma: \gamma \in S^\omega: (\sigma[i] \beta)[k] \gamma \vdash P)))
«because \alpha \vdash A \land \alpha \vdash B = \alpha \vdash (A \lor B)»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash (P \lor (\forall k: 0 \leq k: (\exists \gamma: \gamma \in S^\omega: (\sigma[k] \gamma \vdash P))))
«by definition of Safe(P)»
= (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash Safe(P)))

It is not surprising that Safe(P) is a safety property. If \sigma \# Safe(P) then, by definition, \sigma \# M_P. However, this means there exists an i such that
(\forall \beta: \beta \in S^\omega: \sigma[i] \beta \vdash P).

We could consider prefix \sigma[i] to be a "bad thing". Thus, \sigma violates a safety property whenever \sigma \# Safe(P).

We now show formally that Live(P) satisfies definition (3.2) of liveness.

(\forall \alpha: \alpha \in S^*: true)
«since true \vdash A \lor \neg A»
= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha \beta \vdash P) \lor \neg (\exists \beta: \beta \in S^\omega: \alpha \beta \vdash P))
«renaming bound variable \beta to \gamma»
= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha \beta \vdash P) \lor \neg (\exists \gamma: \gamma \in S^\omega: \alpha \gamma \vdash P))
«since \beta is not free in (\exists \gamma: \gamma \in S^\omega: \alpha \gamma \vdash P)»
= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha \beta \vdash P \lor \neg (\exists \gamma: \gamma \in S^\omega: \alpha \gamma \vdash P)))
«by De Morgan's Law»
= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha \beta \vdash P \lor (\forall \gamma : \gamma \in S^\omega: \alpha \gamma \vdash P)))
\(\langle\text{since true } \land A = A\rangle\)

\[= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = P \lor (|\alpha| = |\alpha| \land (\forall \gamma: \gamma \in S^0: \alpha \gamma \# P)))\]

«by substitution, since \((\alpha \beta)[.|\alpha|] = \alpha\)»

\[= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = P \lor ((|i| = |\alpha|)_{\alpha i} \land (\forall \gamma: \gamma \in S^0: (\alpha \beta)[.|i|] \# P)))\]

«by \(\exists\)-Generalization»

\[\Rightarrow (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = P \lor (\exists i: i = |\alpha|: (\forall \gamma: \gamma \in S^0: (\alpha \beta)[.|i|] \# P)))\]

«by Range Widening»

\[\Rightarrow (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = P \lor (\forall i: 0 \leq i: (\forall \gamma: \gamma \in S^0: (\alpha \beta)[.|i|] \# P)))\]

«by definition of \(\alpha \beta = A\) »

\[= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = P \lor \alpha \beta = \neg (\forall i: 0 \leq i: (\exists \gamma: \gamma \in S^0: \alpha \beta[.|i|] \# P)))\]

«because \(\alpha \beta = A \lor \alpha \beta = B = \alpha \beta = (A \lor B)\)»

\[= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = (P \lor \neg (\forall i: 0 \leq i: (\exists \gamma: \gamma \in S^0: \alpha \beta[.|i|] \# P))))\]

«by definition of \(\text{Live}(P)\) »

\[= (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^0: \alpha \beta = \text{Live}(P)))\]

«by Liveness definition (3.2) »

\(\text{Live}(P)\) is liveness.

An informal justification that \(\text{Live}(P)\) is liveness is the following. If \(\sigma \# \text{Live}(P)\) then, by definition, \(\sigma = M_P\). From, \(\sigma = M_P\), we conclude that it always remains possible for some "good thing" (i.e. \(\beta\) in \(M_P\)) to happen. This is the defining characteristic of liveness, so \(\sigma\) violates a liveness property whenever \(\sigma \# \text{Live}(P)\).

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References
