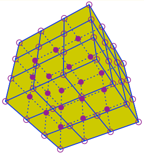

Cumulant Signal Processing, Tensors and some Recurring Problems

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Outline

1. Multilinear Product
2. An "Easy" Problem
3. Symmetric Case
4. Rank r Approximation
5. Displacement Structure

Multilinear Product

Given N matrices $\mathbf{A}^{(n)}$, the N-way (a.k.a. Tucker) product is

$$\mathcal{T}_{j_1,j_2,\ldots,j_N} = \mathbf{A}^{(1)} \star \mathbf{A}^{(2)} \star \cdots \star \mathbf{A}^{(N)} = \sum_{k=1}^{K} \mathbf{A}^{(1)}_{j_1,k} \mathbf{A}^{(2)}_{j_2,k} \cdots \mathbf{A}^{(N)}_{j_N,k}$$

With a core N-dimensional tensor $\mathcal{S}$, the weighted product is

$$\begin{aligned} \mathcal{T} &= \mathbf{A}^{(1)} \overset{\mathcal{S}}{\star} \mathbf{A}^{(2)} \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} \mathbf{A}^{(N)} \\ &= \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} \cdots \sum_{k_N=1}^{K_N} \mathbf{A}^{(1)}_{j_1,k_1} \mathbf{A}^{(2)}_{j_2,k_2} \cdots \mathbf{A}^{(N)}_{j_N,k_N} \mathcal{S}_{k_1,k_2,\ldots,k_N} \end{aligned}$$

and is equivalent to

$$\mathrm{vec}(\mathcal{T}) = \left(\mathbf{A}^{(1)} \otimes \mathbf{A}^{(2)} \otimes \cdots \otimes \mathbf{A}^{(N)}\right) \mathrm{vec}(\mathcal{S})$$

Reduces to unweighted product when $\mathcal{S}_{k_1,k_2,\ldots,k_N} = \delta(k_1, k_2, \ldots, k_N)$.

An "Easy" Approximation Problem

Let $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(N)}$ be a collection of unit-norm column vectors.

Problem

Find unit-norm vectors $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(N)}$ and scalar σ to minimize Frobenius norm of

$$\mathcal{S} - \underbrace{\sigma \cdot \mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \cdots \star \mathbf{u}^{(N)}}_{\text{rank one}}$$

Equivalent problem: Find unit norm vectors to maximize the functional

$$\begin{aligned} f\left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(N)}\right) &= (\mathbf{u}^{(1)})^T \overset{\mathcal{S}}{\star} (\mathbf{u}^{(2)})^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} (\mathbf{u}^{(N)})^T \\ &= \sum_{k_1} \sum_{k_2} \cdots \sum_{k_N} \mathbf{u}^{(1)}_{k_1} \mathbf{u}^{(2)}_{k_2} \cdots \mathbf{u}^{(N)}_{k_N} \mathcal{S}_{k_1, k_2, \ldots, k_N} \end{aligned}$$

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Simple iterative scheme

Can write

$$f = \langle \nabla_1 f, \mathbf{u}^{(1)} \rangle,$$

Thus, given unit-norm vectors $\mathbf{u}^{(1,i)}, \ldots, \mathbf{u}^{(N,i)}$ at iteration i, update $\mathbf{u}^{(1)}$ as

$$\begin{aligned} \mathbf{v} &= \nabla_1 f = \sum_{k_2} \cdots \sum_{k_N} \mathbf{u}^{(2,i)}_{k_2} \cdots \mathbf{u}^{(N,i)}_{k_N} \mathcal{S}_{k_1,k_2,\ldots,k_N} \\ &= \mathbf{I} \overset{\mathcal{S}}{\star} (\mathbf{u}^{(2,i)})^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} (\mathbf{u}^{(N,i)})^T \\ \mathbf{u}^{(1,i+1)} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}$$

and likewise for $\mathbf{u}^{(2)}, \mathbf{u}^{(3)}, \ldots, \mathbf{u}^{(N)}$, and repeat.

Each update gives an increase in $f(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(N)})$, so functional converges monotonically to a local maximum.

Symmetric case

Applications in blind deconvolution, source separation, or independent component analysis have a symmetric tensor:

$$\mathcal{S}_{k_1,k_2,\ldots,k_N} = \text{cum}\left(x_{k_1}, x_{k_2}, \ldots, x_{k_N}\right)$$

One seeks to maximize N-th order cumulant of $\mathbf{u}^T\mathbf{x}$ with $\|\mathbf{u}\| = 1$:

$$f(\mathbf{u}) = \text{cum}_N(\mathbf{u}^T\mathbf{x}) = \underbrace{\mathbf{u}^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} \mathbf{u}^T}_{N \text{ terms}}$$

Symmetric version of previous algorithm gives

$$\mathbf{v} = \sum_{k_2} \cdots \sum_{k_N} \mathbf{u}_{k_2}^{(i)} \cdots \mathbf{u}_{k_N}^{(i)} \mathcal{S}_{k_1,k_2,\ldots,k_N}$$

$$\mathbf{u}^{(i+1)} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Sometimes converges, sometimes not ...

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Sometimes converges, sometimes not ...

Relax unit norm constraint to unit ball:

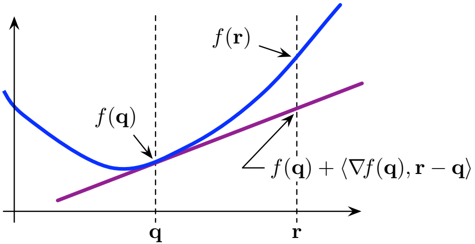
$$\mathcal{B} = \left\{\mathbf{u} : \|\mathbf{u}\| \leq 1\right\}.$$

Special case: $f(\mathbf{u}) = \mathbf{u}^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} \mathbf{u}^T$ is convex (or concave) over $\mathcal{B}$.

Gradient inequality:

$f(\mathbf{r}) \geq \mathbf{f}(\mathbf{q}) + \langle \mathbf{r} - \mathbf{q}, \nabla f(\mathbf{q}) \rangle$

for all $\mathbf{q}, \mathbf{r} \in \mathcal{B}$.



This applies, therefore, to successive iterates: $\mathbf{q} = \mathbf{u}^{(i)}$ and $\mathbf{r} = \mathbf{u}^{(i+1)}$.

From gradient inequality,

$$f(\mathbf{u}^{(i+1)}) - f(\mathbf{u}^{(i)}) \geq \langle \mathbf{u}^{(i+1)} - \mathbf{u}^{(i)}, \nabla f(\mathbf{u}^{(i)}) \rangle$$

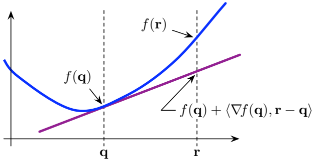
and since $\mathbf{u}^{(i+1)} = \nabla f / \|\nabla f\|$, we have

$$\langle \mathbf{u}^{(i+1)}, \nabla f(\mathbf{u}^{(i)}) \rangle = \|\nabla f(\mathbf{u}^{(i)})\| > \langle \mathbf{u}^{(i)}, \nabla f(\mathbf{u}^{(i)}) \rangle, \qquad \text{if } \mathbf{u}^{(i+1)} \neq \mathbf{u}^{(i)},$$

so that $f(\mathbf{u}^{(i+1)}) > f(\mathbf{u}^{(i)})$.

Remark: The discrepancy

$$D_f(\mathbf{r}, \mathbf{q}) = f(\mathbf{r}) - f(\mathbf{q}) - \langle \mathbf{r} - \mathbf{q}, \nabla f(\mathbf{q}) \rangle \geq 0$$

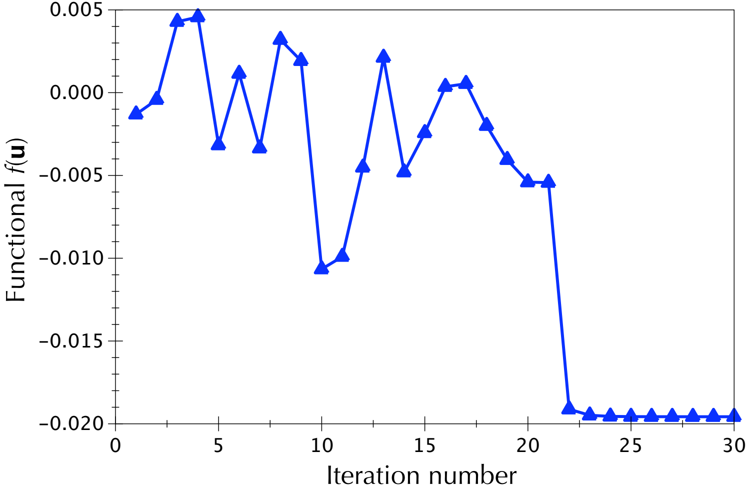


is the Bregman distance induced by f. If f is strongly convex, i.e.,

$$D_f(\mathbf{r}, \mathbf{q}) \geq \alpha \|\mathbf{r} - \mathbf{q}\|^2, \qquad \text{for all } \mathbf{q}, \mathbf{r} \in \mathcal{B},$$

then local convergence rate is at least linear.

What if $f(\mathbf{u})$ is not convex over $\mathcal{B}$?



Gradient interpretation

Rewrite the functional as a "Rayleigh quotient" over $\mathcal{B}$:

$$J(\mathbf{u}) = \frac{f(\mathbf{u})}{\|\mathbf{u}\|^N}, \qquad \mathbf{u} \in \mathcal{B}\backslash\mathbf{0}$$

and consider a gradient ascent algorithm:

$$\begin{aligned} \mathbf{v} &= \mathbf{u}^{(i)} + \mu_i \nabla J(\mathbf{u}^{(i)}) \\ &= \mathbf{u}^{(i)} + \mu_i [\nabla f(\mathbf{u}^{(i)}) - \beta_i \mathbf{u}^{(i)}] \\ \mathbf{u}^{(i+1)} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}$$

with $\beta_i = \langle \nabla f(\mathbf{u}^{(i)}), \mathbf{u}^{(i)} \rangle$.

Previous algorithm is obtained using $\mu_i = 1/\beta_i$.

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A simple "trick"

Consider "regularized" functional

$$g(\mathbf{u}) = f(\mathbf{u}) + \frac{\gamma}{2}\left(\mathbf{u}^T\mathbf{u} - 1\right), \qquad \mathbf{u} \in \mathcal{B},$$

with γ a free parameter, so that $g(\mathbf{u}) = f(\mathbf{u})$ for $\mathbf{u} \in \partial\mathcal{B}$.

Now,

$$g(\mathbf{u}) \text{ is convex over } \mathcal{B} \quad \Leftrightarrow \quad \nabla^2 g(\mathbf{u}) \geq 0 \text{ over } \mathcal{B}.$$

The Hessians are related as

$$\nabla^2 g(\mathbf{u}) = \nabla^2 f(\mathbf{u}) + \gamma\mathbf{I}$$

so with

$$\lambda_* \triangleq \min_{\mathbf{u}\in\mathcal{B}} \left[\lambda_{\min}\left(\nabla^2 f(\mathbf{u})\right)\right]$$

choose $\gamma \geq -\lambda_*$.

Gradient ascent algorithm for "regularized" functional is no different:

$$\begin{aligned} \mathbf{v} &= \mathbf{u}^{(i)} + \mu_i \left[\nabla g_i - \alpha_i \mathbf{u}^{(i)}\right] \qquad \alpha_i = \langle \nabla g_i, \mathbf{u}^{(i)} \rangle \\ &= \mathbf{u}^{(i)} + \mu_i \left[\nabla f_i + \gamma \mathbf{u}^{(i)} - \beta_i \mathbf{u}^{(i)} - \gamma \mathbf{u}^{(i)}\right] \\ &= \mathbf{u}^{(i)} + \mu_i \left[\nabla f_i - \beta_i \mathbf{u}^{(i)}\right] \\ \mathbf{u}^{(i+1)} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}$$

This gives $f(\mathbf{u}^{(i+1)}) > f(\mathbf{u}^{(i)})$ if $\mu_i = 1/\alpha_i = 1/(\beta_i + \gamma)$.

Theorem (R. & Kofidis, 2003)

If $\mathbf{u}^{(i)}$ is not a stationary point, then $f(\mathbf{u}^{(i+1)}) > f(\mathbf{u}^{(i)})$ whenever

$$0 < \mu_i < \frac{2(\beta_i - \lambda_*)}{\beta_i^2 + (\beta_i - \lambda_*)^2 - \|\nabla f(\mathbf{u}^{(i)})\|^2}$$

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&= \mathbf{u}^{(i)} + \mu_i \left[\nabla f_i + \gamma\, \mathbf{u}^{(i)} - \beta_i \mathbf{u}^{(i)} - \gamma\, \mathbf{u}^{(i)}\right] \\
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PARAFAC Decomposition

When a (symmetric or not) tensor *can* be Tucker factored as

$$\mathcal{S} = \mathbf{A}^{(1)} \star \mathbf{A}^{(2)} \star \cdots \star \mathbf{A}^{(N)}$$

uniqueness is addressed by Kruskal ($N = 3$; 1977) and Bro & Sidiropoulos ($N > 3$; 1998), using "k-rank" concepts. (Not always "robust").

Rank-r Approximation: Find rank r matrices $\mathbf{U}^{(i)}$, with $[\mathbf{U}^{(i)}]^T\mathbf{U}^{(i)} = \mathbf{I}_r$, and "core" tensor $\mathcal{T}$ (of size $r \times r \times \cdots \times r$), to minimize Frobenius norm of

$$\mathcal{S} - \mathbf{U}^{(1)} \overset{\mathcal{T}}{\star} \mathbf{U}^{(2)} \overset{\mathcal{T}}{\star} \cdots \overset{\mathcal{T}}{\star} \mathbf{U}^{(N)}$$

- In matrix case, same as r applications of rank-one problem.
- In tensor case, no longer true, unless $\mathcal{T}$ happens to be diagonal.
- In particular, not same as "truncated tensor SVD".

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Symmetric Rank-r Approximation

When $\mathcal{S}$ is symmetric, of dimensions $d \times d \times \cdots d$, introduce tensor

$$\mathcal{R}^{(i)} = \mathbf{I} \overset{\mathcal{S}}{\star} \underbrace{[\mathbf{U}^{(i)}]^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} [\mathbf{U}^{(i)}]^T}_{N-1 \text{ terms}} \qquad (\text{size } d \times r \times \cdots \times r)$$

Its unfolded matrix is (of size $d \times r^{N-1}$)

$$\mathbf{R}^{(i)}_{j,k} = \mathcal{R}^{(i)}_{j,k_2,\ldots,k_N} \qquad \text{with} \qquad k = k_2\, r + k_3\, r^2 + \cdots + k_N\, r^{N-1}$$

Then choose for $\mathbf{U}^{(i+1)}$ (size $d \times r$) the r dominant left singular vectors of $\mathbf{R}^{(i)}$, and iterate.

Convergence

With

$$\mathcal{T} = \underbrace{\mathbf{U}^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} \mathbf{U}^T}_{N \text{ terms}}$$

Can show that

$$\mathcal{T}^{(i+1)} > \mathcal{T}^{(i)}$$

in the sense that

$$\mathbf{v}^T \overset{\mathcal{T}^{(i+1)}}{\star} \cdots \overset{\mathcal{T}^{(i+1)}}{\star} \mathbf{v}^T > \mathbf{v}^T \overset{\mathcal{T}^{(i)}}{\star} \cdots \overset{\mathcal{T}^{(i)}}{\star} \mathbf{v}^T, \quad \forall \mathbf{v} \neq \mathbf{0}$$

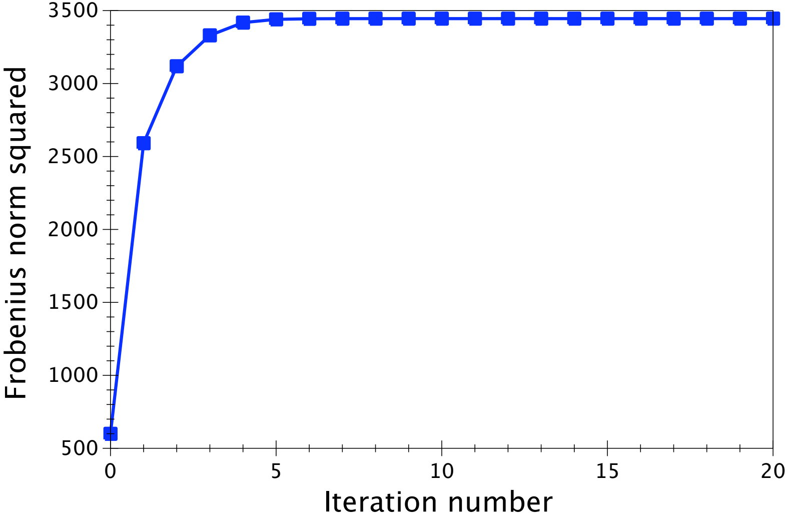
provided

$$f(\mathbf{u}) = \mathbf{u}^T \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} \mathbf{u}^T$$

is convex for $\mathbf{u} \in \mathcal{B}$.

Typical simulation example

$N = 4$, $d = 11$, $r = 5$:



A stationary stochastic process: $r_{|k-l|} = E = (x_{i-k}\, x_{i-l}) = E(x_i\, x_{i+k-l})$ can be modeled as a *linear process*, i.e., $x_i = \sum_j h_j\, \epsilon_{i-j}$ with $\{\epsilon_i\}$ i.i.d, iff

$$\mathbf{R}_N = \begin{bmatrix} r_0 & r_1 & \cdots & r_N \\ r_1 & r_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1 \\ r_N & \cdots & r_1 & r_0 \end{bmatrix} > 0, \quad \text{for all } N$$

Displacement structure:

$$\mathbf{R}_N - \mathbf{Z}\mathbf{R}_N\mathbf{Z}^T = \begin{bmatrix} r_0 & r_1 & \cdots & r_N \\ r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_N & 0 & \cdots & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sqrt{r_0} \\ r_1/\sqrt{r_0} \\ \vdots \\ r_N\sqrt{r_0} \end{bmatrix}}_{\mathbf{a}} [\cdot]^T - \underbrace{\begin{bmatrix} 0 \\ r_1/\sqrt{r_0} \\ \vdots \\ r_N\sqrt{r_0} \end{bmatrix}}_{\mathbf{b}} [\cdot]^T$$

using shift matrix

$$\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & \ddots & \ddots \\ 0 & 1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

Generator formulation: With z_1 and z_2 two complex variables

$$[1 \;\; z_1 \;\; z_1^2 \;\; \cdots] \mathbf{R}_\infty \begin{bmatrix} 1 \\ z_2 \\ z_2^2 \\ \vdots \end{bmatrix} = \frac{a(z_1)\, a(z_2) - b(z_1)\, b(z_2)}{1 - z_1\, z_2}$$

and $\mathbf{R}_\infty \geq 0$ if and only if there exists a Schur function $S(z)$ for which

$$b(z) = S(z)\, a(z)$$

Interpolation problem: Given n points $\{z_i\}$ in unit disk, find $S(z)$ such that

$$b(z_i) = S(z_i)\, a(z_i), \qquad i = 1, 2, \ldots, n.$$

Underlies spectral factorization:

$$H(z) = \frac{z\, Q(z)}{1 - z\, S(z)}, \qquad \text{with} \quad Q(z)\, Q(z^{-1}) + S(z)\, S(z^{-1}) = 1,$$

gives linear model $H(z) = \sum_{k \geq 1} h_k\, z^k$ which generates the correlation data.

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Consider higher order tensor

$$\mathcal{R}_{j_1, j_2 \ldots, j_N} = \text{cum}_N(x_{i-j_1}, \ldots, x_{i-j_N})$$

which is "Toeplitz" whenever $\{x_i\}$ is stationary. Process is "linear" if we can find factorization

$$\mathcal{R} = \gamma_N \underbrace{\mathbf{H} \star \mathbf{H} \star \cdots \star \mathbf{H}}_{N \text{ terms}} \qquad \text{with} \quad \mathbf{H} = \begin{bmatrix} 0 & h_1 & h_2 & h_3 & \cdots \\ 0 & 0 & h_1 & h_2 & \ddots \\ 0 & 0 & 0 & h_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Linear or not, the displacement structure

$$\mathcal{S} \triangleq \mathcal{R} - \mathbf{Z} \overset{\mathcal{R}}{\star} \mathbf{Z} \overset{\mathcal{R}}{\star} \cdots \overset{\mathcal{R}}{\star} \mathbf{Z}$$

vanishes except along its faces.

Consider Hankel matrix

$$\mathbf{\Gamma} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \iddots \\ h_3 & h_4 & h_5 & \iddots \\ \vdots & \iddots & \iddots & \iddots \end{bmatrix} \qquad \Leftrightarrow \qquad H(z) = \sum_{k=1}^{\infty} h_k \, z^k$$

and the identity

$$\mathcal{G} \triangleq \underbrace{\mathbf{\Gamma} \star \cdots \star \mathbf{\Gamma}}_{N \text{ terms}} = \underbrace{\mathbf{H} \star \cdots \star \mathbf{H}}_{\mathcal{R}/\gamma_N} - \mathbf{H}^T \star \cdots \star \mathbf{H}^T$$

$\mathcal{G}$ and $\mathcal{R}/\gamma_N$ agree on faces, and:

$$\mathcal{G} - \mathbf{Z} \overset{\mathcal{G}}{\star} \cdots \overset{\mathcal{G}}{\star} \mathbf{Z} = \mathcal{S}/\gamma_N - \mathbf{h} \star \cdots \star \mathbf{h}$$

with $\mathbf{h} = [0 \;\; h_1 \;\; h_2 \;\; h_3 \;\; \cdots]^T$.

Let $\mathbf{z}_i = [1 \;\; z_i \;\; z_i^2 \;\; z_i^3 \;\; \cdots]$, and introduce generator function

$$S(z_1, z_2, \ldots, z_N) \triangleq \mathbf{z}_1 \overset{\mathcal{S}}{\star} \mathbf{z}_2 \overset{\mathcal{S}}{\star} \cdots \overset{\mathcal{S}}{\star} \mathbf{z}_N$$

Setting $z_N = 1/(z_1 \cdots z_{N-1})$ gives the polyspectrum:

$$\begin{aligned} &S(z_1, \ldots, z_{N-1}, (z_1 \cdots z_{N-1})^{-1}) \\ &= \sum_{k_1, \ldots, k_{N-1}} \mathrm{cum}_N(x_n, x_{n-k_1}, \ldots, x_{n-k_{N-1}})\, z_1^{k_1} \cdots z_{N-1}^{k_{N-1}} \quad \text{(general case)} \\ &= \gamma_N\, H(z_1) \cdots H(z^{N-1})\, H\Big((z_1 \cdots z_{N-1})^{-1}\Big) \quad \text{(when linear)} \end{aligned}$$

Open problem: Given specific evaluation points $\{\alpha_i\}$ in unit disk, when can we find an $H(z)$ whose output cumulants replicate the values[1]

$$S(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_N}) \quad ?$$

[1] V. S. Grigorascu and P. R., "Tensor displacement structures…" in *Fast Algorithms for Matrices w/ Structure*, T. Kailath & A Sayed, eds., SIAM, 1999

Concluding remarks

- Rank one tensor approximation follows easily from matrix case, *except* that it need not lead to a "fundamental" decomposition.
- Symmetric version is convergent if a relaxed function is convex or concave.
- If not, reduce step size (bound obtained analytically).
- Higher-order factorization/interpolation not always so well defined, and offers open terrain.