THE STRONG STABILITY OF ALGORITHMS FOR SOLVING
SYMMETRIC LINEAR SYSTEMS*

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Abstract. An algorithm for solving linear equations is stable on the class of nonsingular symmetric matrices
if the class of symmetric positive definite matrices if the computed solution solves a system that is near the
original problem. Here it is shown that any stable algorithm is also strongly stable on the same matrix class if
the computed solution solves a nearby problem that is also symmetric or symmetric positive definite.

Key words. stability, symmetric matrices

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1. Introduction. When applied to a linear system $Ax = b$, a stable algorithm for
solving systems of linear equations produces a computed solution $\hat{x}$ that is the solution
to a nearby system

$$A\hat{x} = \hat{b},$$

where $\|A - A\|/\|A\|$ is small and $\|\hat{b} - b\|/\|b\|$ is small, for some norm $\|\|$. How "small"
is small enough depends on the accuracy desired in the solution (and on the condition
number of $A$) [15, pp. 189-191]. A proof of the stability of an algorithm usually involves
showing that $\|A - A\|/\|A\|$ and $\|\hat{b} - b\|/\|b\|$ are bounded by $p(n)u$, where $p$ is a low
degree polynomial, $n$ is the order of $A$ and $u$ is the unit roundoff (machine precision).
We would like $p(n)u << 1$.

In solving structured linear equations, it is often important that the perturbed matrix
$A$ have the same structure as $A$. For example, solving electrical network problems gives
rise to symmetric systems of linear equations, $Ax = b$. If the computed solution $\hat{x}$ to
$Ax = b$ satisfies $A\hat{x} = \hat{b}$, but $A$ is not symmetric, then the system $A\hat{x} = \hat{b}$ could never
have arisen from an electrical network problem. But if $A$ is symmetric, then we hope
that there is an electrical network near our original network that gives rise to the system
$A\hat{x} = \hat{b}$.

Another situation where it is important that the perturbed matrix remain symmetric
is in the analysis of Algorithm 5 in [3]. That algorithm uses a variation of inverse iteration
to find the eigenvectors of a certain class of symmetric matrices to high accuracy. The
class includes all symmetric positive definite matrices that can be consistently ordered.
The error analysis uses a new perturbation theorem about symmetric perturbations of
symmetric matrices, and to apply it one needs to know that a nearby symmetric matrix
exists which exactly satisfies the equations at each step of inverse iteration.

The term strongly stable, developed in [4], is used in this context. An algorithm for
solving linear equations is strongly stable for a class of matrices $A$ if for each $A$ in $A$
and for arbitrary \( h \) the computed solution \( \hat{x} \) solves a nearly system \( \hat{A} \hat{x} = \hat{b} \) with \( \hat{A} \) in \( A \). Note that for stability we do not require \( A \) to be in \( A \), but for strong stability we do. (Other stability concepts were introduced in \([12, 13, 14]\).)

In \([4]\) it is shown that the following algorithms for solving linear equations are strongly stable for their respective classes of matrices:

1. Gaussian elimination with partial or complete pivoting on \( A = \{ \text{non-singular matrices} \} \) \([16]\);
2. Cholesky on \( A = \{ \text{symmetric positive definite matrices} \} \) \([16]\);
3. LDL\(^T\) (symmetric Gaussian elimination) on \( A = \{ \text{symmetric matrices} \} \) \([16]\);
4. Symmetric indefinite algorithm (diagonal pivoting method \([5, 6, 9]\)) on \( A = \{ \text{symmetric positive definite matrices} \} \);
5. \( LU \) decomposition (Gaussian eliminations without pivoting) on \( A = \{ \text{strictly column diagonally dominant matrices} \} \) \( (|a_{ii}| > \sum_{j \neq i} |a_{ij}| \text{ for all } i) \) or \( A = \{ \text{strictly column diagonally dominant band matrices} \} \). (See Appendix.)
6. Gaussian elimination with partial or complete pivoting followed by iterative refinement on \( A = \{ \text{non-singular matrices with an arbitrary but fixed sparsity pattern and which are not too ill conditioned} \} \). (See \([12, 13, 14]\) for discussion.)

In \([4]\) it was noted that while Gaussian elimination with partial pivoting and Gaussian elimination with complete pivoting are stable on \( A \) and \( A \) and Aasen's method \([1]\), \([10]\) is stable on \( A \), it does not follow from their error analyses that these algorithms are strongly stable. Thus, the strong stability of these algorithms on \( A \) and \( A \), respectively, was left as an open question.

Here we will extend the list of strongly stable "situations" developed in \([4]\). In particular, we show that \( E \) a method is stable for the class of non-symmetric matrices or the class of symmetric positive definite matrices, then it is strongly stable for the same class.

2. Constructing a symmetric perturbed system. If \( A = A^T \), \( (A + E)z = b, z \neq 0 \), where \( E \) might be nonsymmetric, then we shall construct \( \hat{F} = F^T \) such that \( (A + \hat{F})z = b \) and \( \|F\| \) is within a small constant of \( \|E\| \) for the 2-norm and the Frobenius norm. We shall do this in two different ways. The first will use the Powell-Symmetric-Brayton (PSB) update \([12]\); the second will use a construction via the \( QR \) decomposition; in either case we shall show that \( z \) is the exact solution of a symmetric perturbed system. We include both since the analyses are instructive in their own right.

The problem of nearly symmetric systems has already been addressed in the theory for quasi-Newton methods. For the first approach we shall use the following \([7, 8, \text{p.196}]\).

**Theorem 1.** If \( H \) is symmetric, \( u \neq 0 \), then the unique solution to

\[
\text{minimize } \|H - H_u x - H u^T \|_2
\]

is given by the PSB-update:

\[
H = H_u + \frac{<(y - H_u z)^T + s, (y - H_u z) s^T>}{<s, s>^T} s^T.
\]

Here, \( \| \cdot \| \) is the Frobenius norm and \( <u, v> = u^T v \). We will use this to prove the following theorem.

**Theorem 2.** If \( A = A^T \), \( (A + E)z = b, r = a - Az, z \neq 0 \), then

\[
\hat{F} = rz^T + zr^T = \frac{<(z, z)>}{<(z, z)>} z^T z
\]
satisfies \( (A + \tilde{F})z = b, \tilde{F} = F^T, \) and \( \|F\| \leq 3 \|E\| \) for the 2-norm and the Frobenius norm. Furthermore, \( \tilde{F} \) is the unique solution to

\[
\min \{ \|F\| : F = F^T, (A + F)z = b \}.
\]

**Proof.** In Theorem 1, take \( \hat{H} = A, \hat{z} = z, \hat{y} = b. \) Then \( y = Hz = b - Az = r. \) Thus, the unique \( \tilde{F} \) minimizing \( \{ \|F\| : (A + F)z = b, F = F^T \} \) is the PSB update

\[
\tilde{F} = rz^T + rz^T \quad (rz)^T, 2rz^T.
\]

Thus,

\[
\|F\|_2 \leq \|F\|_2 \leq \frac{\|rz\|_F + \|rz\|_F}{2} \quad \frac{\|rz\|_F + \|rz\|_F}{2rz^T}.
\]

But

\[
\|uv^T\|_2 = \|uw^T\|_2 \leq [u^T w] [v^T w], 10. (p. 16).
\]

Hence,

\[
\|F\|_2 \leq \|F\|_2 \leq \frac{2\|z\|_2 [z; z]}{2\|z\|_2 [z; z]} = \frac{\|z\|_2 [z; z]}{\|z\|_2 [z; z]} = \frac{\|z\|_2 [z; z]}{\|z\|_2 [z; z]},
\]

However, \( r = b - Az = Ez, \) so \( [z; z] \leq \|E\| [z; z]. \) Thus,

\[
\|F\|_2 \leq \|F\|_2 \leq 3Ez \leq 3Ez, \]

Now, we shall construct a symmetric perturbed system by an approach via the QR decomposition which will give a slightly sharper bound. But first we need the following lemma.

**Lemma 1.** Given any two unit vectors \( u \) and \( v, \) there exists a symmetric orthogonal matrix \( P \) such that \( P^T u = v. \)

**Proof.** If \( u \) and \( v \) are parallel, then \( P \) is a multiple of the identity, \( P u = v. \) If \( u \) and \( v \) are not parallel, \( P \) can be taken to be a Householder matrix that reflects in a plane containing \( u \) and \( v \) and is orthogonal to the plane containing \( u \) and \( v. \) \( \square \)

**Theorem 3.** If \( A + \tilde{F}^T, (A + F)z = b, \) \( \neq 0, \) then there exists \( \tilde{F} = F^T \) such that \( (A + F)z = b, \|\tilde{F}\| \leq \|E\| \), and \( \|\tilde{F}\| \leq \|E\|. \) (The bounds are sharp.)

**Proof.** We need to determine \( \tilde{F} \) so that

\[
\tilde{F} = \tilde{F} \text{ and } \tilde{F} = r,
\]

where \( r = b - Az = Ez, \) \( r = 0, \) let \( \tilde{F} = 0. \) Suppose \( r \neq 0. \)

Let \( X = [z/r]^T \text{ is } \text{QR, where } R = [z/r] \tilde{F} = \begin{bmatrix} 0 & \tilde{F} \end{bmatrix} \text{ and } \tilde{F} = \begin{bmatrix} z \tilde{r} \\ 0 \end{bmatrix}. \]

\( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tilde{r} \\ \tilde{r} \end{bmatrix} \) and \( \tilde{r} = \begin{bmatrix} 0 \tilde{r} \end{bmatrix}. \)

be the QR decomposition of \( x. \) Note that, expressing \( \tilde{F} = QFQ^T, \) it is sufficient to determine \( F \) so that

\[
(QQ^T)^{-1}QQF = QFQ^T \quad \text{and } QFQ^T z = \tilde{r},
\]
or, more simply, so that
\[ F^T P = P F \quad \text{and} \quad F E = E F. \]
These can both be satisfied by choosing \( F = \text{diag}(F_1, 0) \) if \( F_1 \) can be determined so that
\[ F_1^T F_1 = I_r \quad \text{and} \quad \begin{bmatrix} F_1^T \xi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ 0 \end{bmatrix}. \]
Since \( \xi \neq 0 \) and \( r \neq 0, \xi_1 \neq 0, \) and \( \xi \neq 0 \) or \( \xi \neq 0, \) let
\[ u = \begin{bmatrix} \xi_1^T \\ 0 \end{bmatrix} = \frac{1}{\|\xi_1\|} \begin{bmatrix} \xi_1 \\ 0 \end{bmatrix}, \]
and
\[ v = \begin{bmatrix} \xi_2^T \\ \xi_3^T \end{bmatrix} / \|\xi_2, \xi_3\|_2. \]
By Lemma 1, there exists \( P = P^T = P^{-1} \) such that \( Pu = v. \) Let \( F_1 = \alpha P, \) where \( \alpha = \|F_2\|/\|E_z\| = \|F_2\|/\|E_z\| \sum \|E_2\|, \) then \( Fz = r \) and \( F^2 = F. \)
If \( \|E_2\| = \|E_2\| \leq \|E_2\| \leq \|E_2\|_2, \) then \( \|E_2\| = \|E_2\| \leq \|F\|, \) and the bound is sharp.
Since \( F_1 \) is a multiple of a \( 2 \times 2 \) orthogonal matrix, \( \|F_1 E\| = \|E_2\| \leq \|F_2\| \leq \|E_2\| \leq \|E_2\|_2. \) Thus.
\[ \|F_2\| = \|E_2\| \leq \|E_2\|_2 \leq \|E_2\| \leq \|E_2\|_2. \]
Seating \( \tilde{F} = QFQ^T \) gives us the result.
However, the \( \tilde{F} \) constructed in Theorem 2 minimizes
\[ (\|F_2\|: F = F, (A + F) \geq b), \]
and, hence
\[ \|F_2\| \leq \|F_2\| \leq \|E_2\|_F. \]
Thus, Theorem 3 gives us the following Corollary.
\textbf{Corollary.} The matrix \( F \) in Theorem 2 satisfies \( \|F\|_F \leq \|E_2\|_F. \)
(Note: In [11] Higham gives a result similar to this Corollary.)

3. Applications. Gaussian elimination with pivoting and Aasen’s method are stable for symmetric systems [10]. But, while the computed solution \( \tilde{x} \) solves a nearby system
\[ (A + E) \tilde{x} = b, \]
it is not the case that the matrix \( E \) is symmetric, at least not from the traditional backward error analyses. Our results show that there is a symmetric \( F \) with \( \|F\|_F \leq \|E\|_F \) and \( \|F\|_F \leq \|E\|_F \) so that
\[ (A + F) \tilde{x} = b. \]
Thus, Gaussian elimination with pivoting and Aasen’s method are strongly stable when applied to symmetric systems. In [4], only the diagonal pivoting method [5], [6], [9] was shown to be strongly stable on symmetric systems. More generally, we have Theorem 4.
THEOREM 4. If a method is stable for nonsingular symmetric matrices, then it is strongly stable for nonsingular symmetric matrices.

Finally we make some observations about strong stability of algorithms for symmetric positive definite systems. The BFGS update [8, p. 201] and the DFP update [8, p. 201] do not give an $F$ near $E$ is the symmetric positive definite case. However, we can make an existence argument as follows.

THEOREM 5. If $A$ is symmetric positive definite and

$$(A + E)x = b$$

with $\|E\|_2 < \lambda_{\min}(A)$, then there exists a symmetric $E$ so that

(1) $$(A + E)x = b,$$

(2) $\|F\|_2 \leq \|E\|_2,$$

and

(3) $$\lambda_{\min}(A + F) > 0.$$  

Proof. Theorem 4 ensures that (1) and (2) hold. From [10, p. 369] or [16, pp. 101–102] we have that

$$\lambda_{\min}(A + F) \geq \lambda_{\min}(A) + \lambda_{\min}(F) \geq \lambda_{\min}(A) - \|F\|_2.$$  

Since $\|E\|_2 < \lambda_{\min}(A)$ and $\|F\|_2 \leq \|E\|_2$, we have $\lambda_{\min}(A + F) > 0$.  

If $A$ is symmetric positive definite, then $\lambda_{\min}(A) = \|A\|_2$. Hence, Theorem 5 says that if $\|E\|_2 < \|A\|_2$, then there exists a symmetric $F$ such that $A + F$ is positive definite, $(A + F)x = b$, and $\|F\|_2 \leq \|E\|_2$.

We shall state this more formally in Theorem 6.

THEOREM 6. If a method is stable for symmetric positive definite matrices, then it is strongly stable for symmetric positive definite matrices.

4. Conclusions. We have shown that any algorithm for linear equations that is stable on $A_1 = \{\text{symmetric positive definite matrices}\}$ or $A_2 = \{\text{nonsingular symmetric matrices}\}$ will also be strongly stable on the same matrix class.

Appendix. A matrix $A$ is strictly column (row) diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for each $i$ (for $|a_{ij}| > \sum_{j \neq i} |a_{ij}|$ for each $i$). Let us perturb $A$ to $A + E$. The following lemma shows that if the perturbation $E$ is small enough then $A + E$ is still strictly column (row) diagonally dominant.

Lemma 2. If $A$ is strictly column (row) diagonally dominant and $\|E\|_i < \delta$, where $\delta = \min_i \{ |a_{ii}| - \sum_{j \neq i} |a_{ij}| \}$ (for each $E_j$), then $A + E$ is strictly column (row) diagonally dominant.

Proof. We shall prove it for column dominance; the proof for row dominance is similar. Since

$$\sum_{j \neq i} |a_{ij} + E_{ij}| \leq \sum_{j \neq i} |a_{ij}| + \sum_{j \neq i} |E_{ij}| < |a_{ii}| - \delta - |e_{ii}|$$

we have

$$\sum_{j \neq i} |a_{ij} + E_{ij}| \leq \sum_{j \neq i} |a_{ij}| + \sum_{j \neq i} |E_{ij}| < |a_{ii}| - \delta - |e_{ii}|$$

$$\leq |a_{ii} + e_{ii}|$$

for each $i$.  

The following theorem shows that if the machine precision \( u \) is small enough then Gaussian elimination without pivoting (LU decomposition) is strongly stable for column strictly diagonally dominant matrices.

**Theorem 7.** Let \( A \) be a column strictly diagonally dominant; let \( z \) be the computed solution by Gaussian elimination without pivoting. Then there exists an \( E \) such that \( (A + E)z = b \), where \( \|E\| \leq p(n)ua \), \( p(n) \) is a low degree polynomial in \( n \), \( u \) is the machine precision, and \( a = \max_{i,j} |a_{ij}| \). If, also, \( u < \beta(p(n)u) \), where \( \beta = \max_{i,j} |a_{ij}| \), then \( A + E \) is strictly column diagonally dominant.

**Proof.** From [10], [15], [16], there is an \( E \) such that \( (A + E)z = b \) with \( \|E\| < p(n)u \max_{i,j} |a_{ij}| \), where \( p \) is a polynomial of degree 3 and \( a_{ij} \) are the elements in the reduced matrices. From [15, Chap. 3], \( \max_{ij} |a_{ij}| \leq 2a \). If \( u < \beta(p(n)u) \), then \( \|E\| < \beta \), and by Lemma 2, \( A + E \) is strictly column diagonally dominant. \( \square \)

**References**