The Kronecker Product

A Product of the Times

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The Kronecker Product

$B \otimes C$ is a *block matrix* whose ij-th block is $b_{ij}C$.

E.g.,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \otimes C = \left[\begin{array}{c|c} b_{11}C & b_{12}C \\ \hline b_{21}C & b_{22}C \end{array}\right]$$

Also called the "Direct Product" or the "Tensor Product"

Every $b_{ij}c_{kl}$ Shows Up

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \otimes \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$=$$

$$\left[\begin{array}{ccc|ccc} b_{11}c_{11} & b_{11}c_{12} & b_{11}c_{13} & b_{12}c_{11} & b_{12}c_{12} & b_{12}c_{13} \\ b_{11}c_{21} & b_{11}c_{22} & b_{11}c_{23} & b_{12}c_{21} & b_{12}c_{22} & b_{12}c_{23} \\ b_{11}c_{31} & b_{11}c_{32} & b_{11}c_{33} & b_{12}c_{31} & b_{12}c_{32} & b_{12}c_{33} \\ \hline b_{21}c_{11} & b_{21}c_{12} & b_{21}c_{13} & b_{22}c_{11} & b_{22}c_{12} & b_{22}c_{13} \\ b_{21}c_{21} & b_{21}c_{22} & b_{21}c_{23} & b_{22}c_{21} & b_{22}c_{22} & b_{22}c_{23} \\ b_{21}c_{31} & b_{21}c_{32} & b_{21}c_{33} & b_{22}c_{31} & b_{22}c_{32} & b_{22}c_{33} \end{array}\right]$$

Basic Algebraic Properties

$$
\begin{aligned}
(B \otimes C)^T &= B^T \otimes C^T \\
(B \otimes C)^{-1} &= B^{-1} \otimes C^{-1} \\
(B \otimes C)(D \otimes F) &= BD \otimes CF \\
B \otimes (C \otimes D) &= (B \otimes C) \otimes D \\
C \otimes B &= (\text{Perfect Shuffle})^T (B \otimes C)(\text{Perfect Shuffle})
\end{aligned}
$$

R.J. Horn and C.R. Johnson(1991). *Topics in Matrix Analysis*, Cambridge University Press, NY.

Reshaping KP Computations

Suppose $B, C \in {\rm I\!R}^{n \times n}$ and $x \in {\rm I\!R}^{n^2}$.

The operation $y = (B \otimes C)x$ is $O(n^4)$:

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y = kron(B,C)*x
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The equivalent, reshaped operation $Y = CXB^T$ is $O(n^3)$:

```
y = reshape(C*reshape(x,n,n)*B',n,n)
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H.V. Henderson and S.R.Searle (1981). "The Vec-Permutation Matrix, The Vec Operator, and Kronecker Products, A Review," *Linear and Mulitilinear Algebra, 9*, 271–288.

Talk Outline

1. **The 1800's**

   Origins: Ⓩ

2. **The 1900's**

   Heightened Profile: $\otimes \; \otimes \; \otimes \; \otimes \; \otimes \; \otimes \; \otimes \; \bigotimes \; \bigotimes \; \bigotimes$

3. **The 2000's**

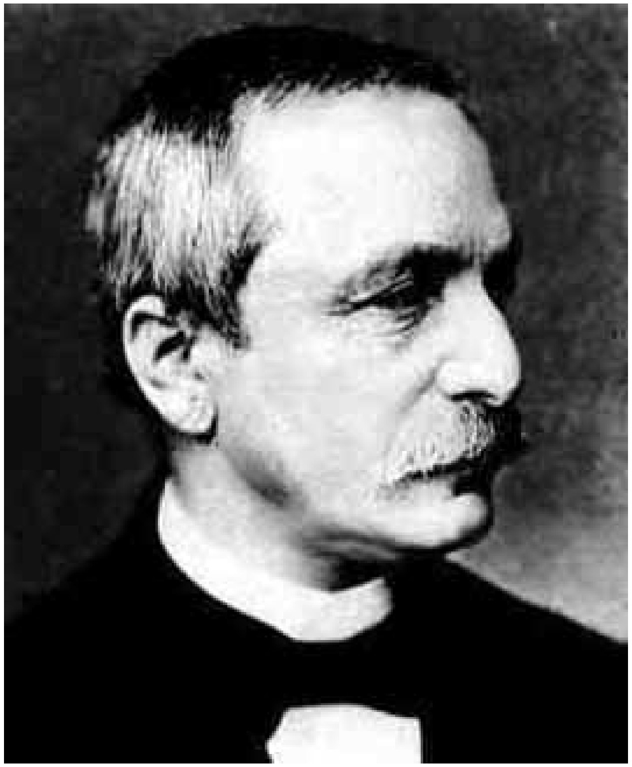
   Future: $\textcircled{\infty}$

The 1800's

Ⓩ

Products and Deltas

$\bigotimes$



δ_{ij}

Leopold Kronecker (1823–1891)

Of course, the contributions go far beyond this...

E.T. Bell (1937). *Men of Mathematics*, Simon and Schuster, New York.

K. Hensel (1968). *Leopold Kronecker's Werke*, Chelsea Publishing Company, New York.

Brief Survey of the Kronecker Delta

$$U^T \delta_{ij} V = |\delta_{ij}|$$

$$\kappa_2(\delta_{ij}) = \frac{1}{\delta_{ij}}$$

G.H. Golub and C. Van Loan (1996). *Matrix Computations, 3rd Ed.*, Johns Hopkins University Press, Baltimore, Maryland.

Acknowledgement

H.V. Henderson, F. Pukelsheim, and S.R. Searle (1983). "On the History of the Kronecker Product," *Linear and Multilinear Algebra 14*, 113–120.

Shayle Searle, Professor Emeritus, Cornell University (right)

Scandal!

H.V. Henderson, F. Pukelsheim, and S.R. Searle (1983). "On the History of the Kronecker Product," *Linear and Multilinear Algebra 14*, 113–120.

Abstract

History reveals that what is today called the Kronecker product should be called the Zehfuss Product.

This fact is somewhat appreciated by the modern (numerical) linear algebra community:

R.J. Horn and C.R. Johnson(1991). *Topics in Matrix Analysis*, Cambridge University Press, NY, p. 254.

A.N. Langville and W.J. Stewart (2004). "The Kronecker product and stochastic automata networks," *J. Computational and Applied Mathematics 167*, 429–447.

Who Was Zehfuss?

Born 1832.

Obscure professor of mathematics at University of Heidelberg for a while. Then went on to other things.

Wrote papers on determinants...

G. Zehfuss (1858). "Über eine gewisse Determinante," *Zeitschrift für Mathematik und Physik* 3, 298–301.

Main Result a.k.a. "The Z Theorem"

If $B \in \mathrm{IR}^{m\times m}$ and $C \in \mathrm{IR}^{n\times n}$ then

$$\det(B \otimes C) \;=\; \det(B)^n \det(C)^m$$

Modern Proof

Note that $I_n \otimes B$ and $I_m \otimes C$ are block diagonal and take determinants in

$$B \otimes C \;=\; (B \otimes I_n)(I_m \otimes C) \;=\; P(I_n \otimes B)P^T(I_m \otimes C)$$

where P is a perfect shuffle permutation.

Excerpts from Zehfuss(1858)

$$\Delta = \begin{vmatrix} a_1\mathcal{A}_1 & a_1\mathcal{B}_1 & b_1\mathcal{A}_1 & b_1\mathcal{B}_1 & c_1\mathcal{A}_1 & c_1\mathcal{B}_1 & d_1\mathcal{A}_1 & d_1\mathcal{B}_1 \\ a_1\mathcal{A}_2 & a_1\mathcal{B}_2 & b_1\mathcal{A}_2 & b_1\mathcal{B}_2 & c_1\mathcal{A}_2 & c_1\mathcal{B}_2 & d_1\mathcal{A}_2 & d_1\mathcal{B}_2 \\ a_2\mathcal{A}_1 & a_2\mathcal{B}_1 & b_2\mathcal{A}_1 & b_2\mathcal{B}_1 & c_2\mathcal{A}_1 & c_2\mathcal{B}_1 & d_2\mathcal{A}_1 & d_2\mathcal{B}_1 \\ a_2\mathcal{A}_2 & a_2\mathcal{B}_2 & b_2\mathcal{A}_2 & b_2\mathcal{B}_2 & c_2\mathcal{A}_2 & c_2\mathcal{B}_2 & d_2\mathcal{A}_2 & d_2\mathcal{B}_2 \\ a_3\mathcal{A}_1 & a_3\mathcal{B}_1 & b_3\mathcal{A}_1 & b_3\mathcal{B}_1 & c_3\mathcal{A}_1 & c_3\mathcal{B}_1 & d_3\mathcal{A}_1 & d_3\mathcal{B}_1 \\ a_3\mathcal{A}_2 & a_3\mathcal{B}_2 & b_3\mathcal{A}_2 & b_3\mathcal{B}_2 & c_3\mathcal{A}_2 & c_3\mathcal{B}_2 & d_3\mathcal{A}_2 & d_3\mathcal{B}_2 \\ a_4\mathcal{A}_1 & a_4\mathcal{B}_1 & b_4\mathcal{A}_1 & b_4\mathcal{B}_1 & c_4\mathcal{A}_1 & c_4\mathcal{B}_1 & d_4\mathcal{A}_1 & d_4\mathcal{B}_1 \\ a_4\mathcal{A}_2 & a_4\mathcal{B}_2 & b_4\mathcal{A}_2 & b_4\mathcal{B}_2 & c_4\mathcal{A}_2 & c_4\mathcal{B}_2 & d_4\mathcal{A}_2 & d_4\mathcal{B}_2 \end{vmatrix}$$

Excerpts from Zehfuss(1858)

$$p = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \qquad \text{und} \qquad P = \begin{vmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{vmatrix}$$

$$\Delta_{2\,2,2} = p_4^2 P_2^4$$

$$\Delta_{2\,Mm} = p^M P^m$$

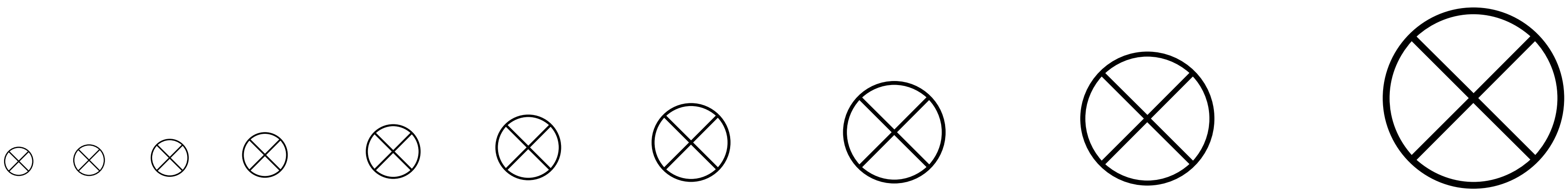
Hensel (1891)

Student in Berlin 1880-1884.

Maintains that Kronecker presented the Z-theorem in his lectures.

K. Hensel (1891). "Über die Darstellung der Determinante eines Systems, welches aus zwei anderen componirt ist," *ACTA Mathematica 14*, 317–319.

The 1900's



Muir (1911)

Attributes the Z-theorem to Zehfuss.

Calls $\det(B \otimes C)$ the "Zehfuss determinant."

T. Muir (1911). *The Theory of Determinants in the Historical Order of Development, Vols 1-4*, Dover, NY.

Rutherford(1933)

Q. When are two Zehfuss matrices equal?

$$B \otimes C \overset{???}{=} F \otimes G$$

Subscripting from zero, if B $(m_b \times n_b)$, C $(m_c \times n_c)$, F $(m_f \times n_f)$, G $(m_g \times n_g)$, then $(B \otimes C)_{ij} = (F \otimes G)_{ij}$ means

$$B(\text{floor}(i/m_c), \text{floor}(j/n_c)) \cdot C(i \bmod m_c, j \bmod n_c)$$

$$=$$

$$F(\text{floor}(i/m_g), \text{floor}(j/n_g)) \cdot G(i \bmod m_g, j \bmod n_g)$$

D.E. Rutherford (1933). "On the Condition that Two Zehfuss Matrices are Equal," *Bull. Amer. Math. Soc. 39*, 801-808.

$\textcircled{\mathbf{Z}} \longrightarrow \bigotimes$ **Why?**

"...a series of influential texts at and after the turn of the century permanently associated Kronecker's name with the "$\otimes$" product and this terminology is nearly universal today."

Horn and Johnson (1991)

"...the textbook of Scott and Matthews (1904) which appeared four years after the publication of Rados' paper, gave new life to the old error. This was probably due to the teaching of Pascal, whose second edition (1923) still propagates the error [of the first edition (1897).]"

Muir (1927)

Heightened Profile Beginning in the 60s

Some Reasons

Regular Grids

Tensoring Low Dimension Ideas

Higher Order Statistics

Fast Transforms

Preconditioners

Quantum Computing

Tensor Decompositions/Approximations

C. Van Loan (2000). "The Ubiquitous Kronecker Product," *Journal of Computational and Applied Mathematics*,, 85-100.

Regular Grids

$(M+1)$-by-$(N+1)$ discretization of the Laplacian on a rectangle...

$$A \;=\; I_M \otimes T_N \;+\; T_M \otimes I_N$$

$$T_5 \;=\; \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

F.W. Dorr (1970). "The Direct Solution of the Discrete Poisson Equation on a Rectangle," *SIAM Review 12*, 248–263.

G.H. Golub and C.F. Van Loan (1996). *Matrix Computations, 3rd Ed*, Johns Hopkins University Press, Baltimore, MD.

Tensoring Low Dimension Ideas

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i\, f(x_i) \;=\; w^T f(x)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,dx\,dy\,dz \approx \sum_{i=1}^{n_x}\sum_{j=1}^{n_y}\sum_{k=1}^{n_z} w_i^{(x)} w_j^{(y)} w_k^{(z)} f(x_i, y_j, z_k)$$

$$= (w^{(x)} \otimes w^{(y)} \otimes w^{(z)})^T f(x \otimes y \otimes z)$$

A. Graham (1981). *Kronecker Products and Matrix Calculus with Applications*, Ellis Horwood Ltd, Chichester, England.

Higher Order Statistics

$$\mathrm{E}(xx^T)$$
$$\Downarrow$$
$$\mathrm{E}(x \otimes x)$$
$$\Downarrow$$
$$\mathrm{E}(x \otimes x \otimes \cdots \otimes x)$$

Kronecker powers:

$$\otimes^k A \;=\; A \otimes A \otimes \cdots \otimes A \qquad\qquad (k \text{ times})$$

T.F. Andre, R.D. Nowak, and B.D. Van Veen (1997). "Low Rank Estimation of Higher Order Statistics," *IEEE Trans. Signal Processing 45*, 673–685.

Fast Transforms

FFT

$$F_{16}P_{16} \;=\; B_{16}(I_2 \otimes \; B_8)(I_4 \otimes B_4)(I_8 \otimes B_2)$$

$$B_4 \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega_4 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega_4 \end{bmatrix} \qquad \omega_n \;=\; \exp(-2\pi i/n)$$

J. Granata, M. Conner, and R. Tolimieri (1992). "Recursive Fast Algorithms and the Role of the Tensor Product," *IEEE Transactions on Signal Processing 40*, 2921–2930.

C. Van Loan(1992). *Computational Frameworks for the Fast Fourier Transform*, SIAM Publications, Philadelphia, PA.

Fast Transforms Cont'd

Haar Wavelet Transform

$$W_{2m} = \begin{cases} \left[W_m \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| I_m \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] & \text{if } m > 1 \\ [\,1\,] & \text{if } m = 1 \end{cases}.$$

Fast Gauss Transform

$$g_{ij} = \exp(-\| s_j - t_i \|_2^2 / \delta) \Rightarrow G = G_{near} + G_{far}$$

G_{near} involves a Kronecker Product

G. Strang(1993). "Wavelet Transforms Versus Fourier Transforms," *Bulletin of the American Mathematical Association, 28*, 288–305.

X. Sun and Y. Bao (2003). "A Kronecker Product Represenation of the Fast Gauss Transform," *SIAM J. Matrix Anal. Appl., 24*, 768–786.

Preconditioners

If $A \approx B \otimes C$, then $B \otimes C$ has potential as a preconditioner.

It captures the essence of A.

It is easy to solve $(B \otimes C)z = r$.

Good Example: A band block Toeplitz with banded Toeplitz blocks. B and C chosen to be band Toeplitz.

J. Kamm and J.G. Nagy (2000). "Optimal Kronecker Product Approximation of Block Toeplitz Matrices," *SIAM J. Matrix Anal. and Appl., 22*, 155–172.

J. Nagy and M. Kilmer (2006). "Kronecker Product Approximation for Three-Dimensional Imaging Applications," *IEEE Trans. Image Proc. 15*, 604-613.

Quantum Computing

Filled with Kronecker powers of 2-by-2's.

$$H^{\otimes n} = H \otimes H \otimes \cdots \otimes H \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

N.D. Mermin (2007). *Quantum Computer Science*, Cambridge University Press, Cambridge, England.

Tensor Decompositions/Approximation

E.g. Given $\mathcal{A} = \mathcal{A}(1{:}n, 1{:}n, 1{:}n, 1{:}n)$, find orthogonal

$$Q = \left[\, q_1 \;\; \cdots \;\; q_n \,\right]$$

$$U = \left[\, u_1 \;\; \cdots \;\; u_n \,\right]$$

$$V = \left[\, v_1 \;\; \cdots \;\; v_n \,\right]$$

$$W = \left[\, w_1 \;\; \cdots \;\; w_n \,\right]$$

and a "core tensor" σ so

$$\text{vec}(\mathcal{A}) \;\approx\; \sum_{i,j,k,\ell=1}^{n} \sigma_{ijk,\ell}\, w_i \otimes v_j \otimes u_k \otimes q_\ell$$

T. Kolda and B. Bader (2009). "Tensor Decompositions and Applications," *SIAM Review 51*, 455–500.

Descendants

1. **The Left Kronecker Product**
2. **The Hadamard Product**
3. **The Tracy-Singh Product**
4. **The Khatri-Rao Product**
5. **The Generalized Kronecker Product**
6. **The Strong Kronecker Product**
7. **The Symmetric Kronecker Product**
8. **The Bi-Alternate Product**

Left Kronecker Product

Definition:

$$B \overset{\text{Left}}{\otimes} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11}B & c_{12}B \\ c_{21}B & c_{22}B \end{bmatrix} = C \otimes B$$

Fact:

If $B \in \mathbb{R}^{m_b \times n_b}$ and $C \in \mathbb{R}^{m_c \times n_c}$ then

$$B \overset{\text{Left}}{\otimes} C \;=\; \Pi^T_{m_c,\, m_b m_c}(B \otimes C)\Pi_{n_c,\, n_b n_c}$$

$\uparrow$ Perfect Shuffles $\uparrow$

F.A. Graybill(1969). *Introduction to Matrices with Applications in Statistics*, Wadsworth, Belmont, CA.

The Hadamard Product

Definition:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \overset{\text{Had}}{\otimes} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} b_{11}c_{11} & b_{12}c_{12} \\ b_{21}c_{21} & b_{22}c_{22} \\ b_{31}c_{31} & b_{32}c_{32} \end{bmatrix}$$

$$B \overset{\text{Had}}{\otimes} C \; = \; B.*C$$

A. Smilde, R. Bro, and P. Geladi (2004). *Multiway Analysis*, John Wiley, Chichester, England.

The Hadamard Product

Fact:

If $\tilde{A} = B \otimes C$ and $B, C \in \mathbb{R}^{m \times n}$, then

$$B \overset{\text{Had}}{\otimes} C = \tilde{A}(1{:}(m+1){:}m^2, 1{:}(n+1){:}n^2)$$

E.g.,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \otimes \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \left[\begin{array}{cc|cc} b_{11}c_{11} & b_{11}c_{12} & b_{12}c_{11} & b_{12}c_{12} \\ b_{11}c_{21} & b_{11}c_{22} & b_{12}c_{21} & b_{12}c_{22} \\ b_{11}c_{31} & b_{11}c_{32} & b_{12}c_{31} & b_{12}c_{32} \\ \hline b_{21}c_{11} & b_{21}c_{12} & b_{22}c_{11} & b_{22}c_{12} \\ b_{21}c_{21} & b_{21}c_{22} & b_{22}c_{21} & b_{22}c_{22} \\ b_{21}c_{31} & b_{21}c_{32} & b_{22}c_{31} & b_{22}c_{32} \\ \hline b_{31}c_{11} & b_{31}c_{12} & b_{32}c_{11} & b_{32}c_{12} \\ b_{31}c_{21} & b_{31}c_{22} & b_{32}c_{21} & b_{32}c_{22} \\ b_{31}c_{31} & b_{31}c_{32} & b_{32}c_{31} & b_{32}c_{32} \end{array}\right]$$

The Tracy-Singh Product

Definition:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$B \overset{\text{TS}}{\otimes} C = \left[\begin{array}{cc|cc} B_{11} \otimes C_{11} & B_{11} \otimes C_{12} & B_{12} \otimes C_{11} & B_{12} \otimes C_{12} \\ B_{11} \otimes C_{21} & B_{11} \otimes C_{22} & B_{12} \otimes C_{21} & B_{12} \otimes C_{22} \\ \hline B_{21} \otimes C_{11} & B_{21} \otimes C_{12} & B_{22} \otimes C_{11} & B_{22} \otimes C_{12} \\ B_{21} \otimes C_{21} & B_{21} \otimes C_{22} & B_{22} \otimes C_{21} & B_{22} \otimes C_{22} \\ \hline B_{31} \otimes C_{11} & B_{31} \otimes C_{12} & B_{32} \otimes C_{11} & B_{32} \otimes C_{12} \\ B_{31} \otimes C_{21} & B_{31} \otimes C_{22} & B_{32} \otimes C_{21} & B_{32} \otimes C_{22} \end{array}\right]$$

D.S. Tracy and R.P. Singh (1972). "A New Matrix Product and Its Applications in Partitioned Matrices," *Statistica Neerlandica 26*, 143–157.

The Khatri-Rao Product

Definition:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}$$

$$B \overset{\text{K-R}}{\otimes} C = \left[\begin{array}{c|c} B_{11} \otimes C_{11} & B_{12} \otimes C_{12} \\ \hline B_{21} \otimes C_{21} & B_{22} \otimes C_{22} \\ \hline B_{31} \otimes C_{31} & B_{32} \otimes C_{32} \end{array}\right]$$

C.R. Rao and S.K. Mitra (1971). *Generalized Inverse of Matrices and Applications*, John Wiley and Sons, New York.

A. Smilde, R. Bro, and P. Geladi (2004). *Multiway Analysis*, John Wiley, Chichester, England.

The Khatri-Rao Product

Fact:

$$B \overset{\text{KR}}{\otimes} C \quad \text{is a submatrix of} \quad B \overset{\text{TS}}{\otimes} C$$

$$B \overset{\text{TS}}{\otimes} C = \left[\begin{array}{cc|cc} \color{red}{B_{11} \otimes C_{11}} & B_{11} \otimes C_{12} & B_{12} \otimes C_{11} & \color{red}{B_{12} \otimes C_{12}} \\ B_{11} \otimes C_{21} & B_{11} \otimes C_{22} & B_{12} \otimes C_{21} & B_{12} \otimes C_{22} \\ B_{11} \otimes C_{31} & B_{11} \otimes C_{32} & B_{12} \otimes C_{31} & B_{12} \otimes C_{32} \\ \hline B_{21} \otimes C_{11} & B_{21} \otimes C_{12} & B_{22} \otimes C_{11} & B_{22} \otimes C_{12} \\ \color{red}{B_{21} \otimes C_{21}} & B_{21} \otimes C_{22} & B_{22} \otimes C_{21} & \color{red}{B_{22} \otimes C_{22}} \\ B_{21} \otimes C_{31} & B_{21} \otimes C_{32} & B_{22} \otimes C_{31} & B_{22} \otimes C_{32} \\ \hline B_{31} \otimes C_{11} & B_{31} \otimes C_{12} & B_{32} \otimes C_{11} & B_{32} \otimes C_{12} \\ B_{31} \otimes C_{21} & B_{31} \otimes C_{22} & B_{32} \otimes C_{21} & B_{32} \otimes C_{22} \\ \color{red}{B_{31} \otimes C_{31}} & B_{31} \otimes C_{32} & B_{32} \otimes C_{31} & \color{red}{B_{32} \otimes C_{32}} \end{array}\right]$$

The Generalized Kronecker Product

$$\left\{\begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array}\right\} \overset{\text{gen}}{\otimes} C \;=\; \left[\begin{array}{c} B_1 \overset{\text{Left}}{\otimes} C(1,:) \\ \\ B_2 \overset{\text{Left}}{\otimes} C(2,:) \\ \\ B_3 \overset{\text{Left}}{\otimes} C(3,:) \\ \\ B_4 \overset{\text{Left}}{\otimes} C(4,:)] \end{array}\right]$$

Regalia and Mitra (1989), "Kronecker Products, Unitary Matrices, and Signal Processing Applications," *SIAM Review, 31,* 586–613.

The Generalized Kronecker Product

$$\left\{\begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array}\right\} \overset{\text{GEN}}{\otimes} \left\{\begin{array}{c} C_1 \\ C_2 \end{array}\right\} = \left\{\begin{array}{c} \left\{\begin{array}{c} B_1 \\ B_2 \end{array}\right\} \overset{\text{gen}}{\otimes} C_1 \\ \\ \left\{\begin{array}{c} B_3 \\ B_4 \end{array}\right\} \overset{\text{gen}}{\otimes} C_2 \end{array}\right\}$$

The Strong Kronecker Product

A block matrix multiplication, but with Kronecker Products instead matrix-matrix products, e.g.,

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \overset{\text{Strong}}{\otimes} \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

$$=$$

$$\left[\begin{array}{c|c} B_{11} \otimes C_{11} + B_{12} \otimes C_{21} & B_{11} \otimes C_{12} + B_{12} \otimes C_{22} \\ \hline B_{21} \otimes C_{11} + B_{22} \otimes C_{21} & B_{21} \otimes C_{12} + B_{22} \otimes C_{22} \end{array}\right]$$

W. De Launey and J. Seberry (1994), "The Strong Kronecker Product," *Journal of Combinatorial Theory, Series A 66*, 192–213.

The Symmetric Kronecker Product

The KP turns matrix equations into vector equations:

$$CXB^T \;=\; G \qquad \Leftrightarrow \qquad (B \otimes C)\,\mathrm{vec}(X) \;=\; \mathrm{vec}(G)$$

The symmetric Kronecker product does the same thing for matrix equations with symmetric solutions:

$$\tfrac{1}{2}\left(CXB^T + BXC^T\right) \;=\; G \qquad \text{(symmetric)}$$

$$\Leftrightarrow$$

$$(B \overset{\mathrm{sym}}{\otimes} C)\,\mathrm{svec}(X) \;=\; \mathrm{svec}(G)$$

where

$$\mathrm{svec}\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}\right) = \begin{bmatrix} x_{11} & \sqrt{2}x_{12} & x_{22} & \sqrt{2}x_{13} & \sqrt{2}x_{23} & x_{33} \end{bmatrix}^T$$

F. Alizadeh, J-P.A. Haeberly, and M.L. Overton (1998). "Primal-Dual Interior Point Methods for Semidefinite Programming: Convergence Rates, Stability, and Numerical Results," *SIAM J. Optimization 8*, 746–768.

Symmetric Kronecker Product

Fact: If

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & \alpha & 0 & 0 & 1 \end{bmatrix} \qquad \alpha = 1/\sqrt{2}$$

then $\mathrm{vec}(X) = P \cdot \mathrm{svec}(X)$ and

$$B \overset{\mathrm{sym}}{\otimes} C = P^T (B \otimes C) P$$

H.V. Henderson and S.R. Searle(1998). "Vec and Vech Operators for Matrices, with Some uses in Jacobians and Multivariate Statistics," *The Canadian Journal of Statistics 7*, 65–81.

Bi-Alternate Product

$$B \overset{\text{Bi-Alt}}{\otimes} C = \frac{1}{2}(B \otimes C + C \otimes B)$$

W. Govaerts (2000). *Numerical Methods for Bifurcations of Dynamical Equilibria*, SIAM Publications, Philadelhia, PA.

The 2000's

Three Predictions

Big N Will Mean Big d Will Mean KP

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$N = 2^d$$

G. Beylkin and M.J. Mohlenkamp (2005). "Algorithms for Numerical Analysis in High Dimensions," *SIAM J. Scientific Computation 26*, 2133–2159.

Inevitable: Scalar $\rightarrow$ Block $\rightarrow$ Tensor

Tensor-level thinking will require an ability to spot KP's. E.g., if for all $1 \leq m_i \leq n$ we have,

$$\mathcal{B}(m_1, m_2, m_3, m_4)$$

$$=$$

$$\sum_{i_1,i_2,i_3,i_4=1}^{n} W(i_1, m_1)Y(i_2, m_2)X(i_3, m_3)Z(i_4, m_4)\mathcal{A}(i_1, i_2, i_3, i_4)$$

then

$$B \;=\; (W \otimes Y)^T A(X \otimes Z)$$

NSF Workshop on Future Directions in Tensor-Based Computation and Modeling, 2009. `http://www.cs.cornell.edu/cv/TenWork/FinalReport.pdf`.

Data-Sparse Approximate Factorizations

New KP-based factorizations will widen the set of solvable huge problems.

Sample factorization...

$$A \;\approx\; (B_1 \otimes C_1)(B_2 \otimes C_2)(B_3 \otimes C_3)\cdots$$

Det(Log(A)) via Zehfuss

If

$$A \approx (B \otimes C)(D \otimes E)(F \otimes G) \cdots$$

then the big log det problem becomes a bunch of smaller ones...

$$\begin{aligned} \log(\det(A)) \approx \; & n_c \log(\det(B)) + n_b \log(\det(C)) \;\; + \\ & n_e \log(\det(D)) + n_d \log(\det(E)) \;\; + \\ & n_g \log(\det(F)) + n_f \log(\det(G)) \cdots \end{aligned}$$

R.P. Barry and R.K. Pace (1999). "Monte Carlo Estimates of the Log Determinant of Large Sparse Matrices," *Linear Algebra and Its Applications 289*, 41–54.