Understanding Mrs. Divitz

Metaphor and Computational Mathematics

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Householder XVIII: Tahoe City

What Recent Math Test Scores Show...

4th Grade	8th Grade	12th Grade
1. Singapore	1. Singapore	1. Netherlands
2. Korea	2. Korea	2. Sweden
3. Japan	3. Japan	3. Denmark
4. Hong Kong	4. Hong Kong	4. Switzerland
5. Netherlands	5. Belgium	5. Iceland
6. Czech Republic	6. Czech Republic	6. Norway
⋮	⋮	⋮
12. United States	**28.** United States	**19.** United States

A theory to cover the facts: Word Problems

Romy and Michele's High School Reunion

Romy

(Mira Sorvino)

Michele

(Lisa Kudrow)

ROMY AND MICHELE'S
HIGH SCHOOL REUNION

THE Quote

Hey Romy, remember Mrs. Divitz's class?
There was like always a word problem.

THE Quote

Like there's a guy in a rowboat going X miles, and the current is going like, you know, some other miles, and how long does it take him to get to town?

THE Quote

It's like, who cares? Who wants to go to town with a guy who drives a rowboat?

Understanding Mrs. Divitz

How important is it to teach computation by using exciting applications?

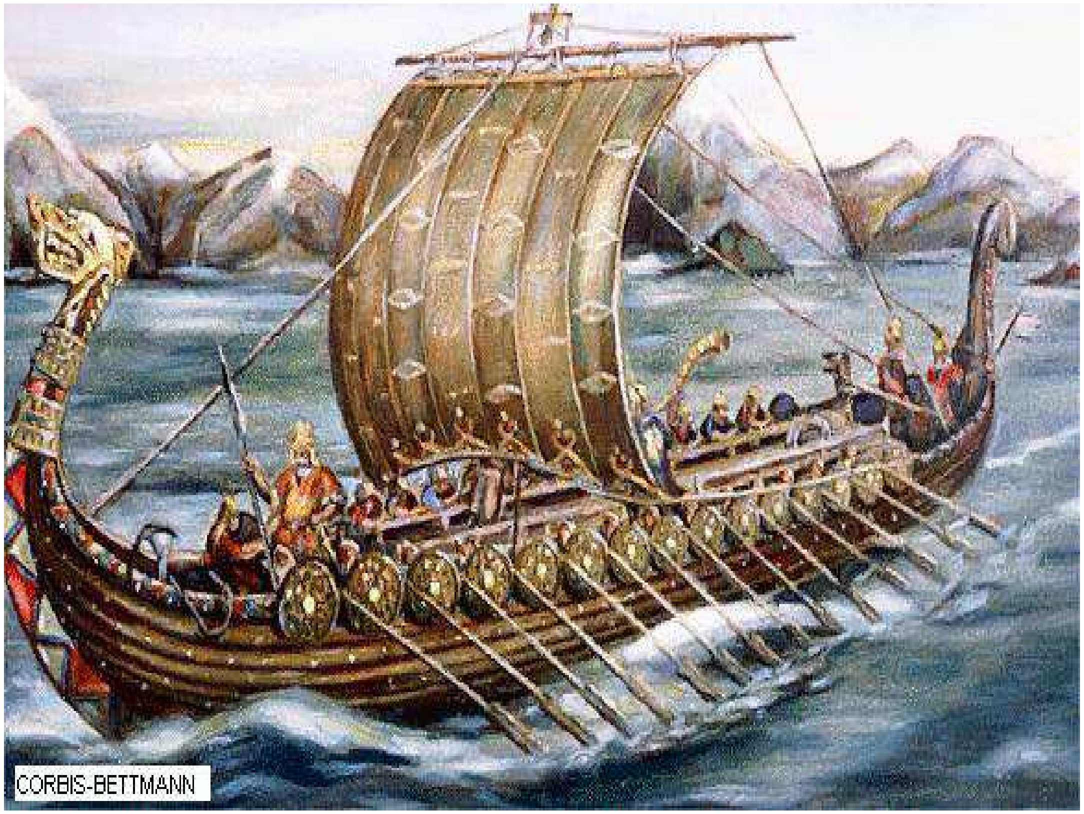
Can a rowboat problem ever be interesting?

Was Mrs. Divitz using the rowboat as a metaphor?

Rowboat Problems Around the World

Different Strokes for Different Folks...

The Scandinavian Approach



Who wants to plunder the north coast of Europe
with somebody to drives a rowboat?

The British Approach

Who wants to read Maths at Oxford or Cambridge
with somebody who crews on a rowboat?

The East Coast Approach

Who wants to join a revolution with somebody who crosses the Delaware in a rowboat?

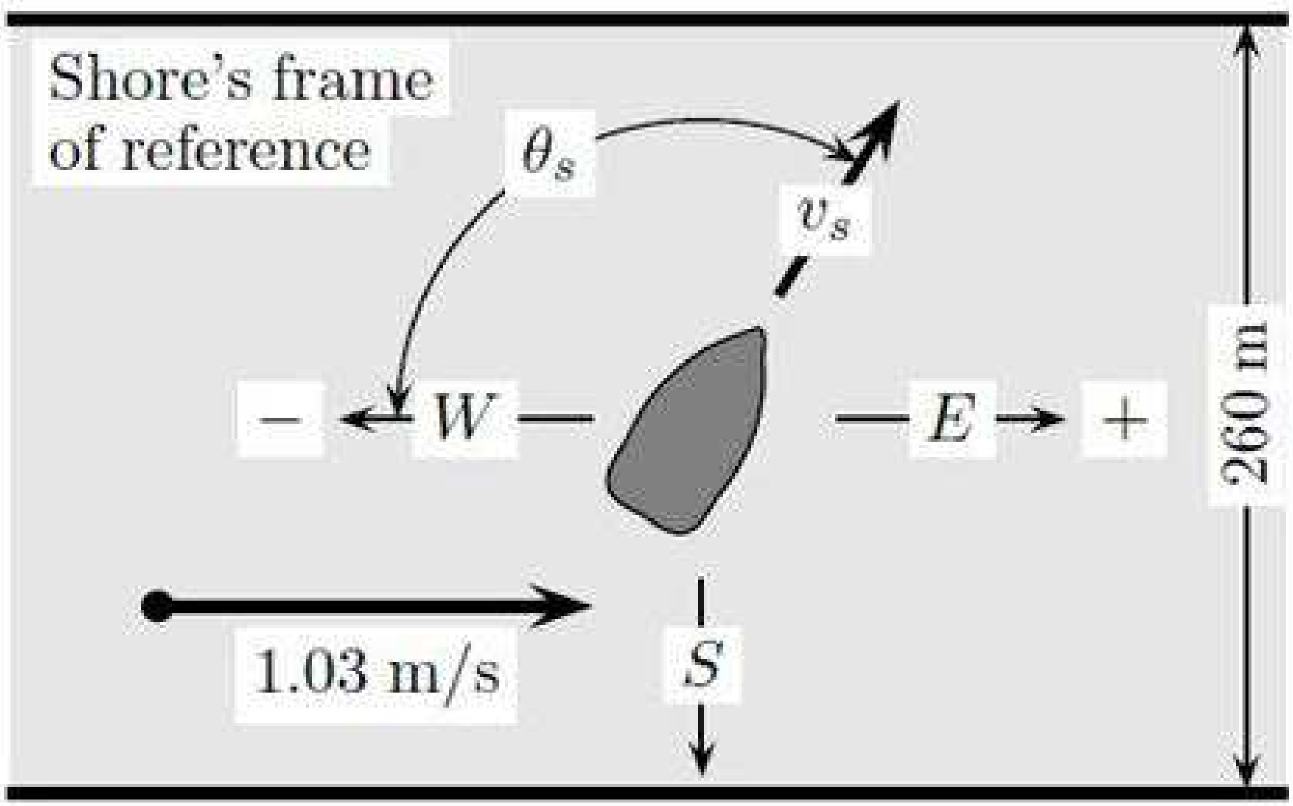
The West Coast Approach

Who heads to Berkeley in the ’60’s with somebody who drives a Volkswagen with a rowboat “option”?

The Householder Approach

Who sets out for Tahoe City with a guy in a Kayak?

Are Rowboat Problems Too Abstract?



It depends...

Matlab Has a Role To Play...

```
  function T = RowBoat(X,Y,Distance)

% X is some miles per hour

% Y is like you know some other miles.

% T is when whats-his-name gets to town.

  T = Distance/(X-Y);
```

Note how easy it is to vectorize this!

Metaphors Also Have a Role to Play

The Worst Lack-of-Rowboat Problem of All Time

Example 1. The Xeno Problem

The Wrong Approach

Prove that if $|r| < 1$ and

$$S_n = r + r^2 + r^3 + \cdots + r^n,$$

then

$$\lim_{n\to\infty} S_n = \frac{r}{1-r}\,.$$

Example 1. The Xeno Problem

The Divitz "Metaphor" Approach

Bob is one meter away from the head of the line at McDonalds. Every minute his distance to the counter is halved. When does Bob get his Big Mac? Show work.

Solution.

Bob never gets his Big Mac...

$$1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64,\ 1/128,\ \ldots$$

Van Loan, C.F. (2011). *The Xeno Diet*, SIAM Publications, Philadelphia.

Example 2. The Missing Data Problem

The Wrong Approach

What is 3×4? What is 3×5? What is 4×4? What is 4×5?

x	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	[redacted]	[redacted]	18	21	24	27	30
4	0	4	8	12	[redacted]	[redacted]	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

MathATube.com Together we'll learn.

Example 2. The Missing Data Problem

The Divitz "Metaphor" Approach

Complete the following matrix so that it has minimum rank:

x	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	[redacted]	[redacted]	18	21	24	27	30
4	0	4	8	12	[redacted]	[redacted]	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

MathATube.com Together we'll learn.

Example 3. The Sudoku Problem

The Wrong Approach

For the following matrix, verify by hand that every row, column, and block is made up of the integers 1 through 9.

2	7	3	9	5	4	1	6	8
9	6	4	1	2	8	3	7	5
5	1	8	3	7	6	2	9	4
7	2	5	4	8	1	9	3	6
8	3	1	5	6	9	7	4	2
4	9	6	2	3	7	5	8	1
1	4	7	8	9	5	6	2	3
3	5	9	6	4	2	8	1	7
6	8	2	7	1	3	4	5	9

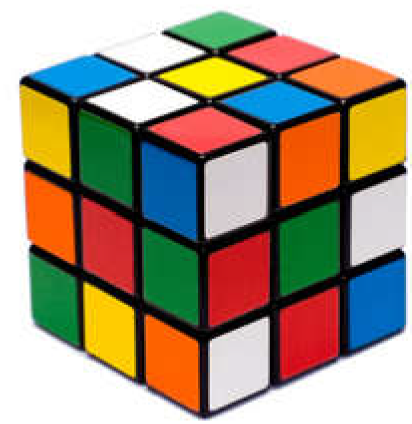
Example 3. The Sudoku Problem

The Divitz "Metaphor" Approach

U^T

2	7	3	9	5	4	1	6	8
9	6	4	1	2	8	3	7	5
5	1	8	3	7	6	2	9	4
7	2	5	4	8	1	9	3	6
8	3	1	5	6	9	7	4	2
4	9	6	2	3	7	5	8	1
1	4	7	8	9	5	6	2	3
3	5	9	6	4	2	8	1	7
6	8	2	7	1	3	4	5	9

$V =$



On to the Feature Film...

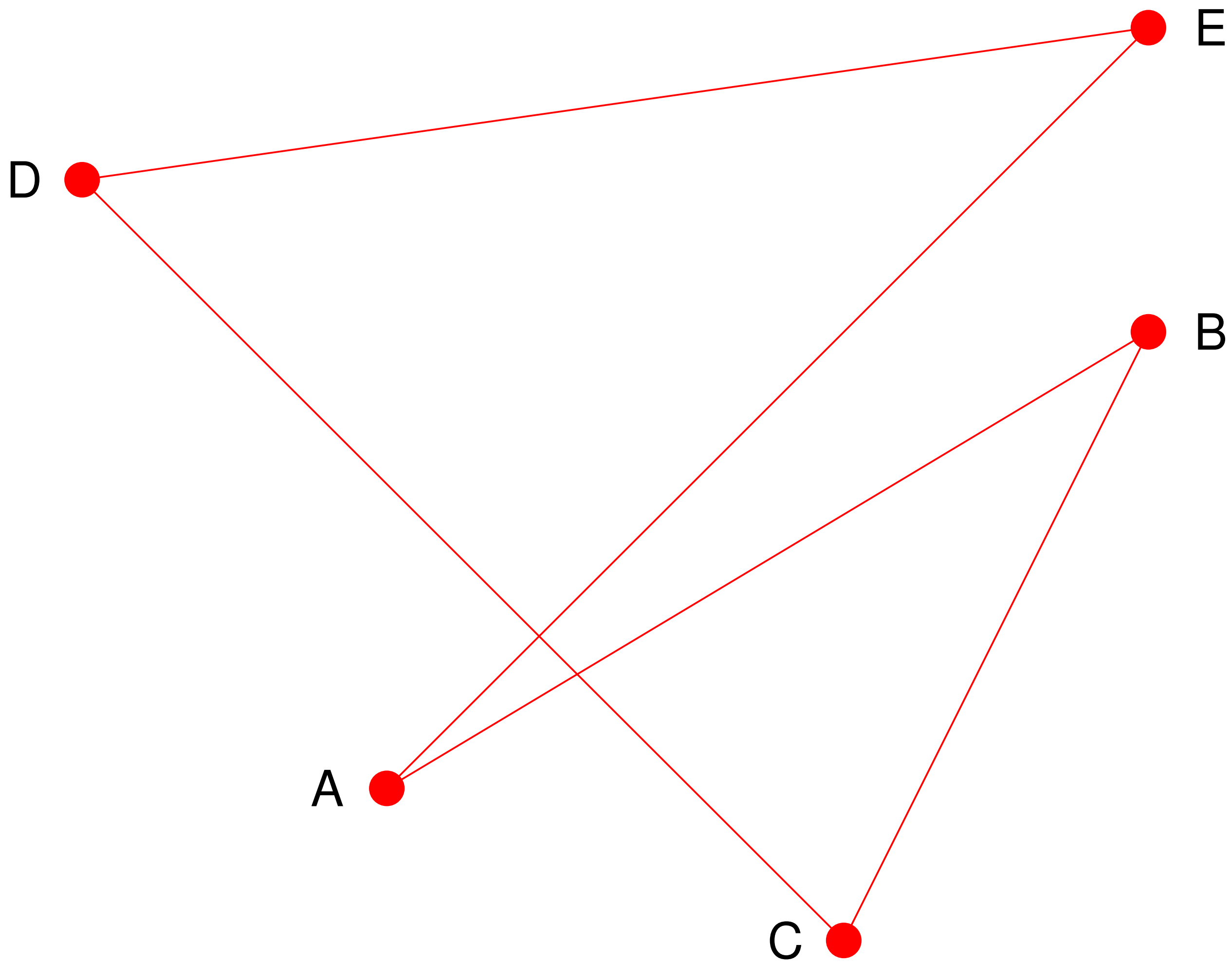
A Freshman Programming Problem

Given a polygon, generate a new polygon by connecting the midpoints of its sides.

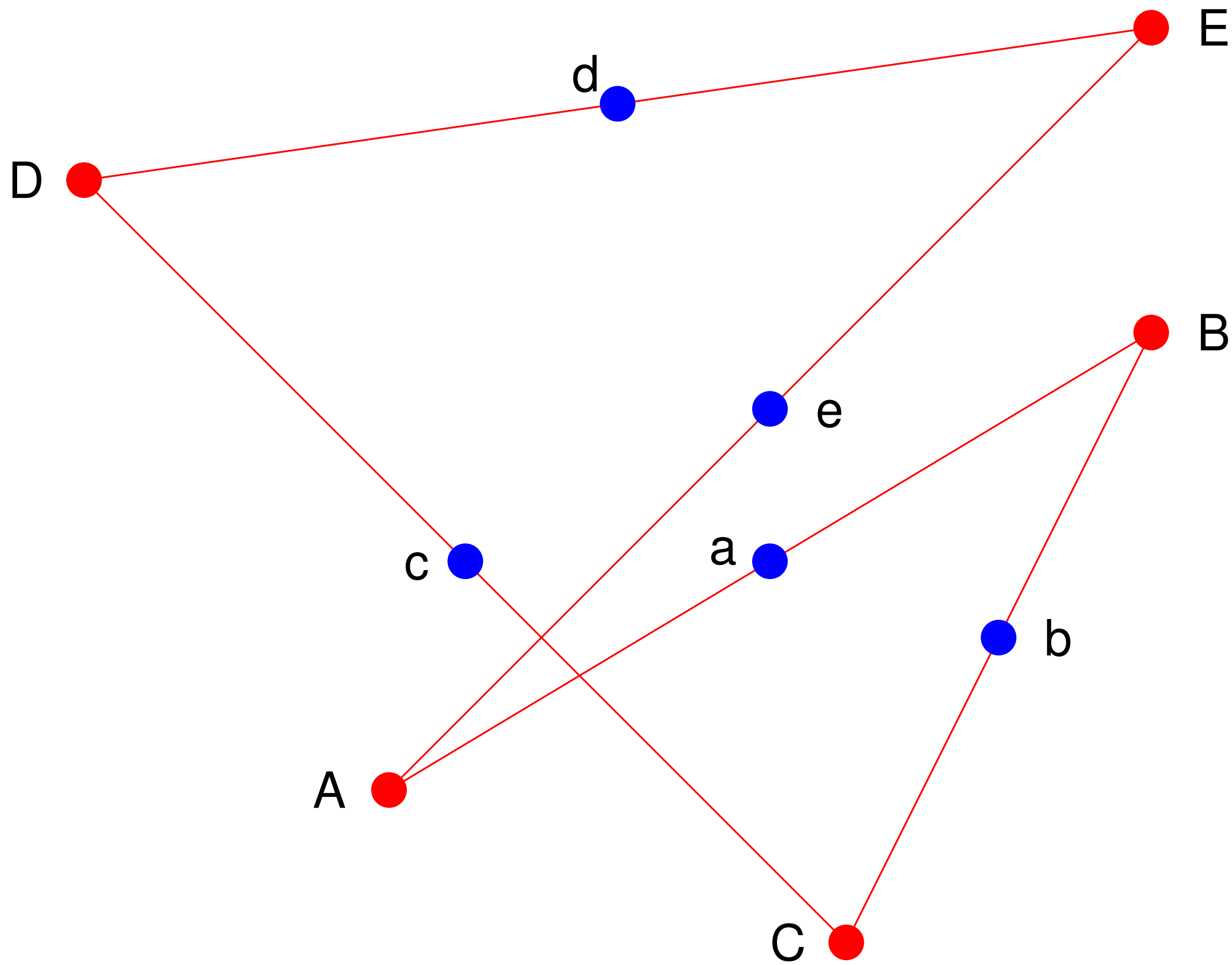
Repeat the process many times and show what happens graphically.

Turns out to be an excellent example of the Divitz "Metaphor Approach".

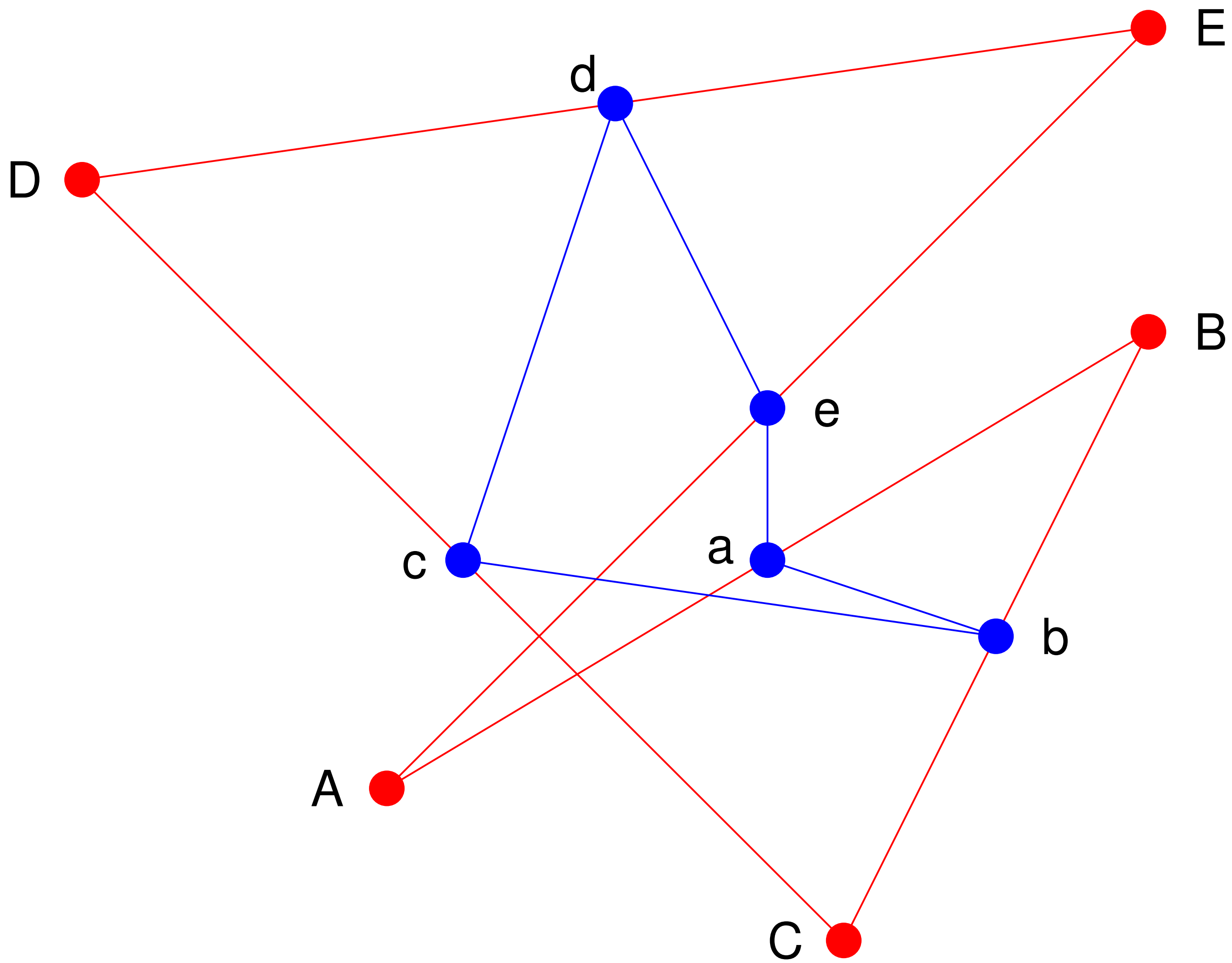
A Polygon..



The Midpoints...



The New Polygon...

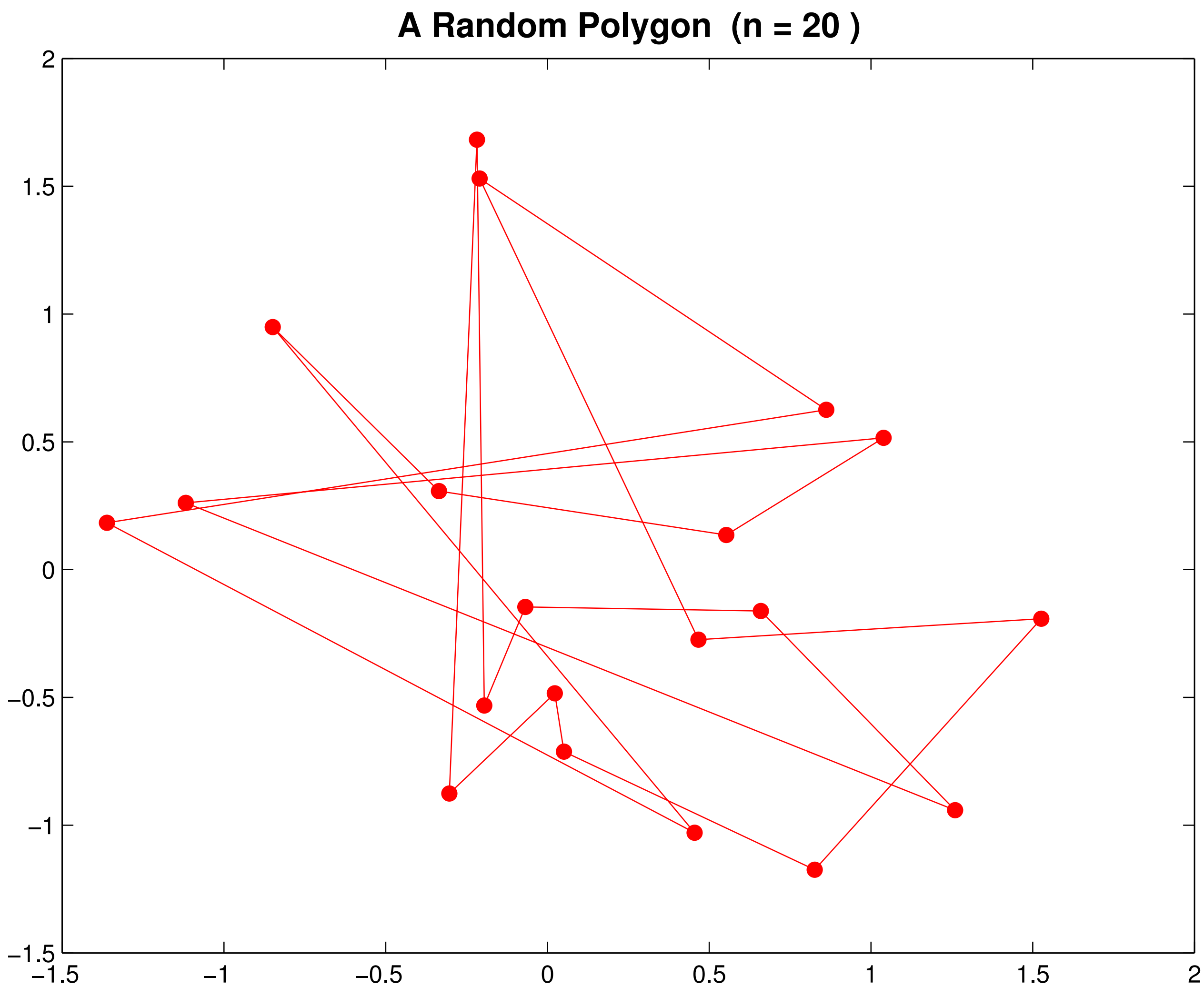


The Average of Two Polygons

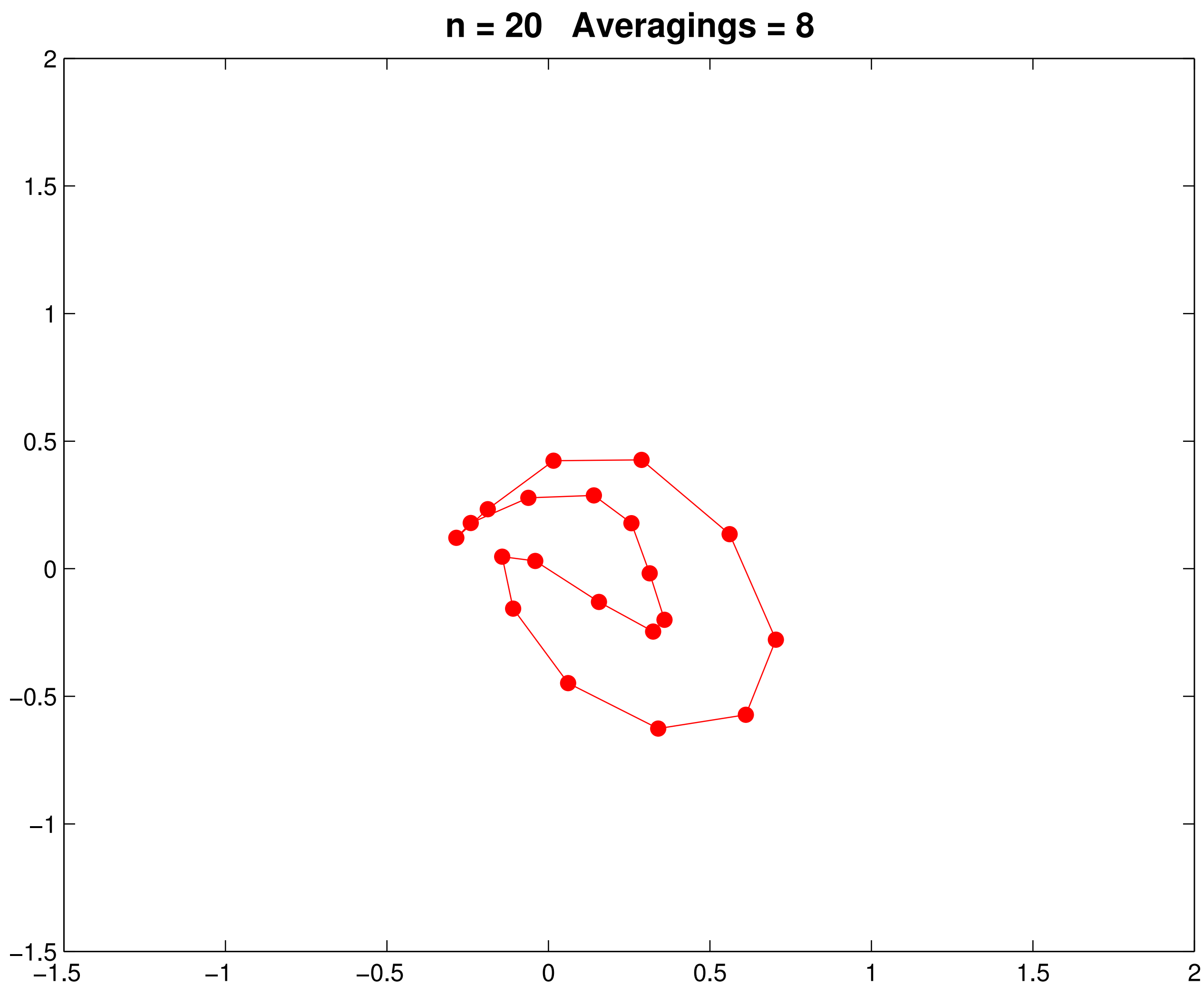
```
     x = [ 0.0  5.0   3.0  -2.0  5.0 ];
xShift = [ 5.0  3.0  -2.0   5.0  0.0 ];
  xNew = [ 2.5  4.0   0.5   1.5  2.5 ];

     y = [ 1.0  4.0   0.0   5.0  6.0 ];
yShift = [ 4.0  0.0   5.0   6.0  1.0 ];
  yNew = [ 2.5  2.0   2.5   5.5  3.5 ];
```

$$\mathcal{P}_{New} = \left(\mathcal{P} + \mathcal{P}_{Shift}\right)/2$$

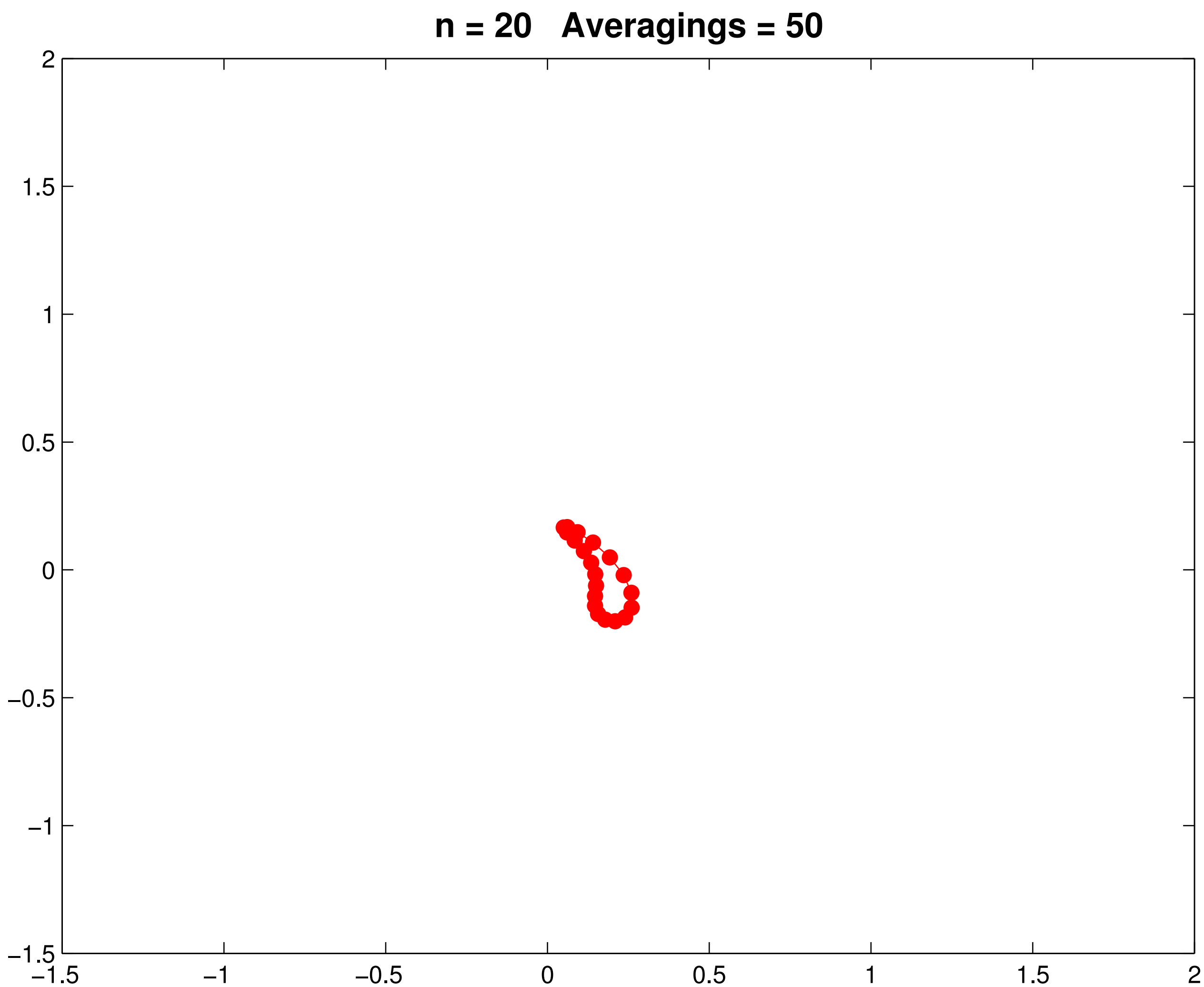
Solution 1



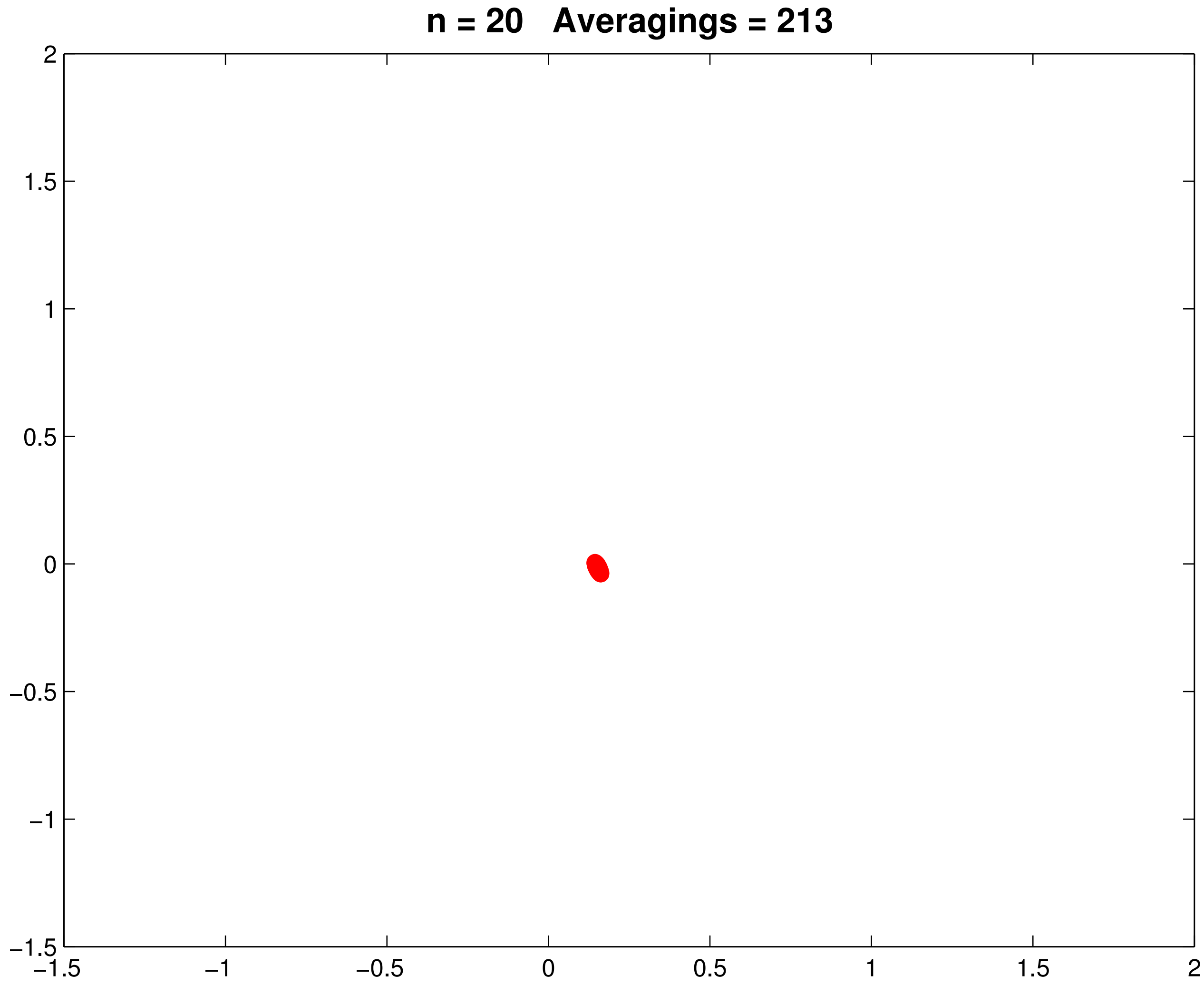
Solution 1



Solution 1



Solution 1

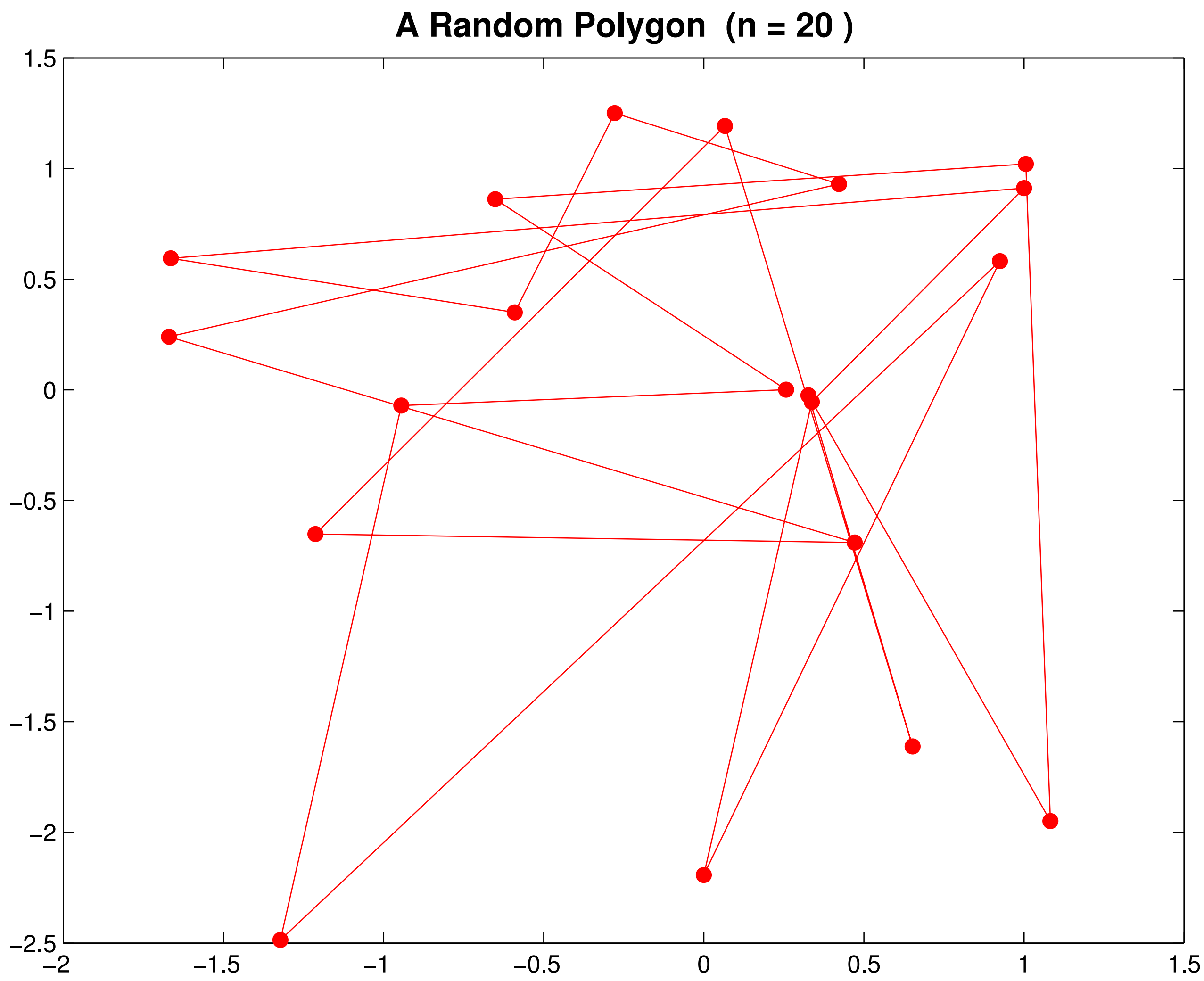


A Freshman Programming Problem (Revised)

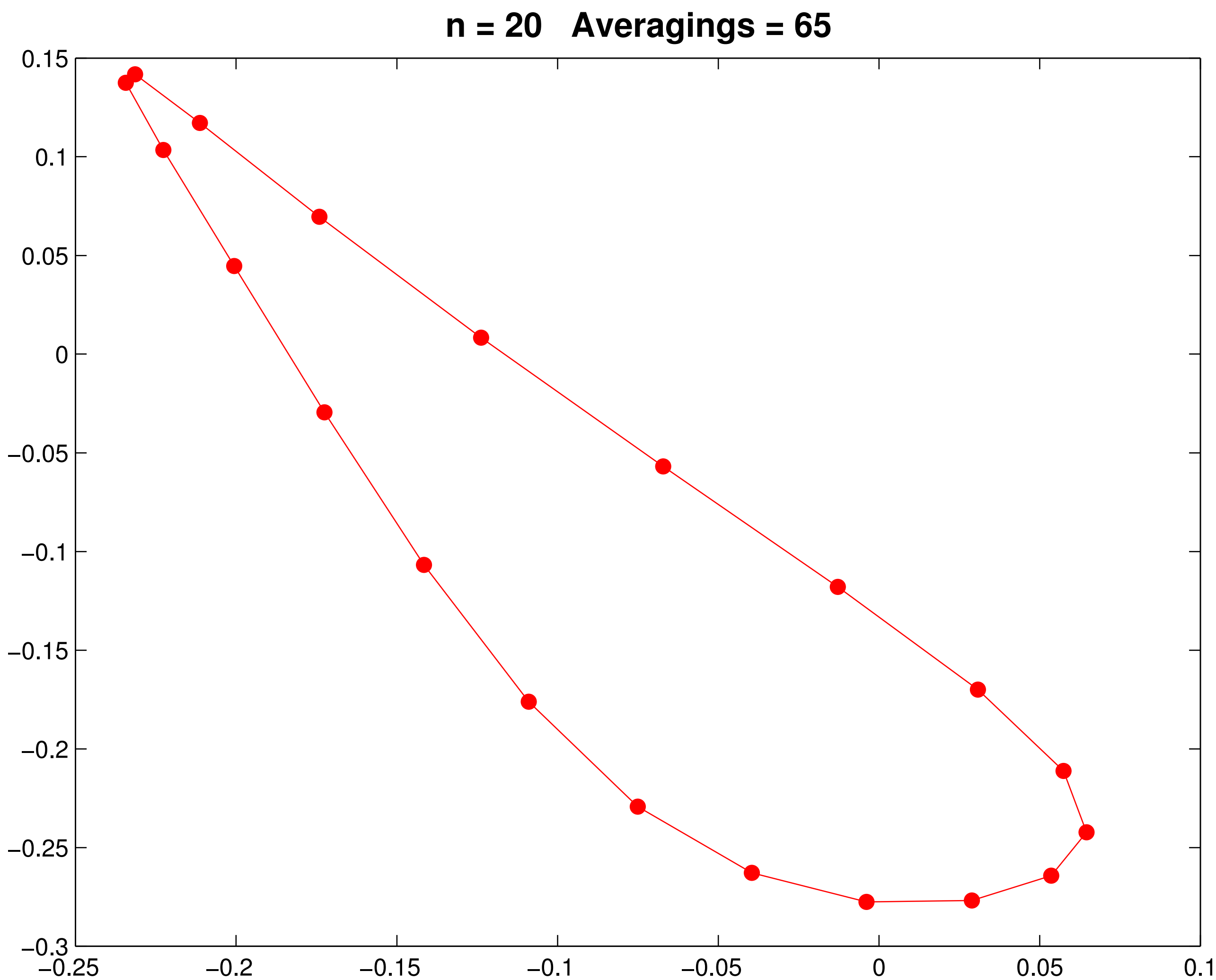
Given a polygon, generate a new polygon by connecting the midpoints of its sides.

Repeat the process many times and show what happens graphically. Use autoscaling.

Solution 2

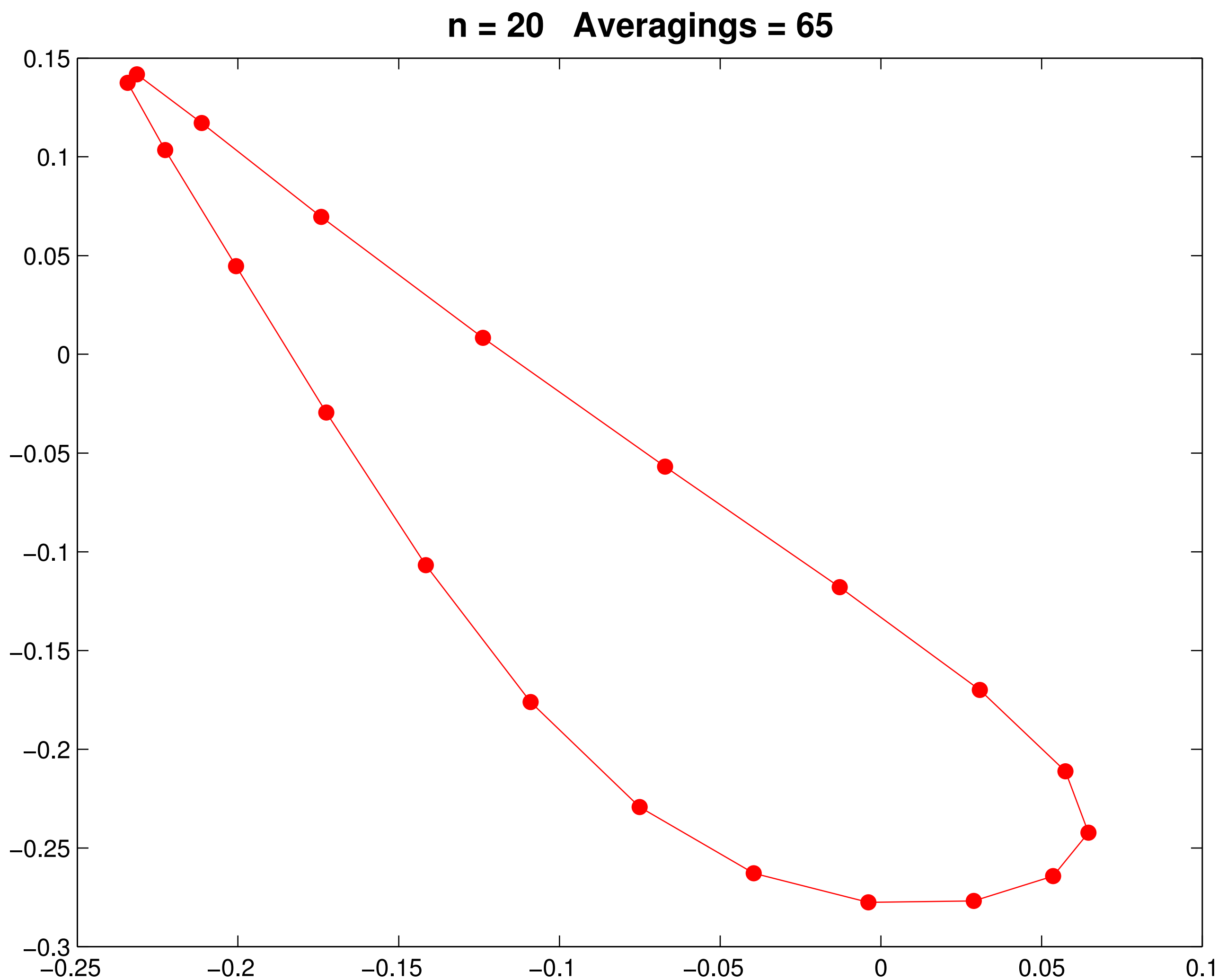


Solution 2



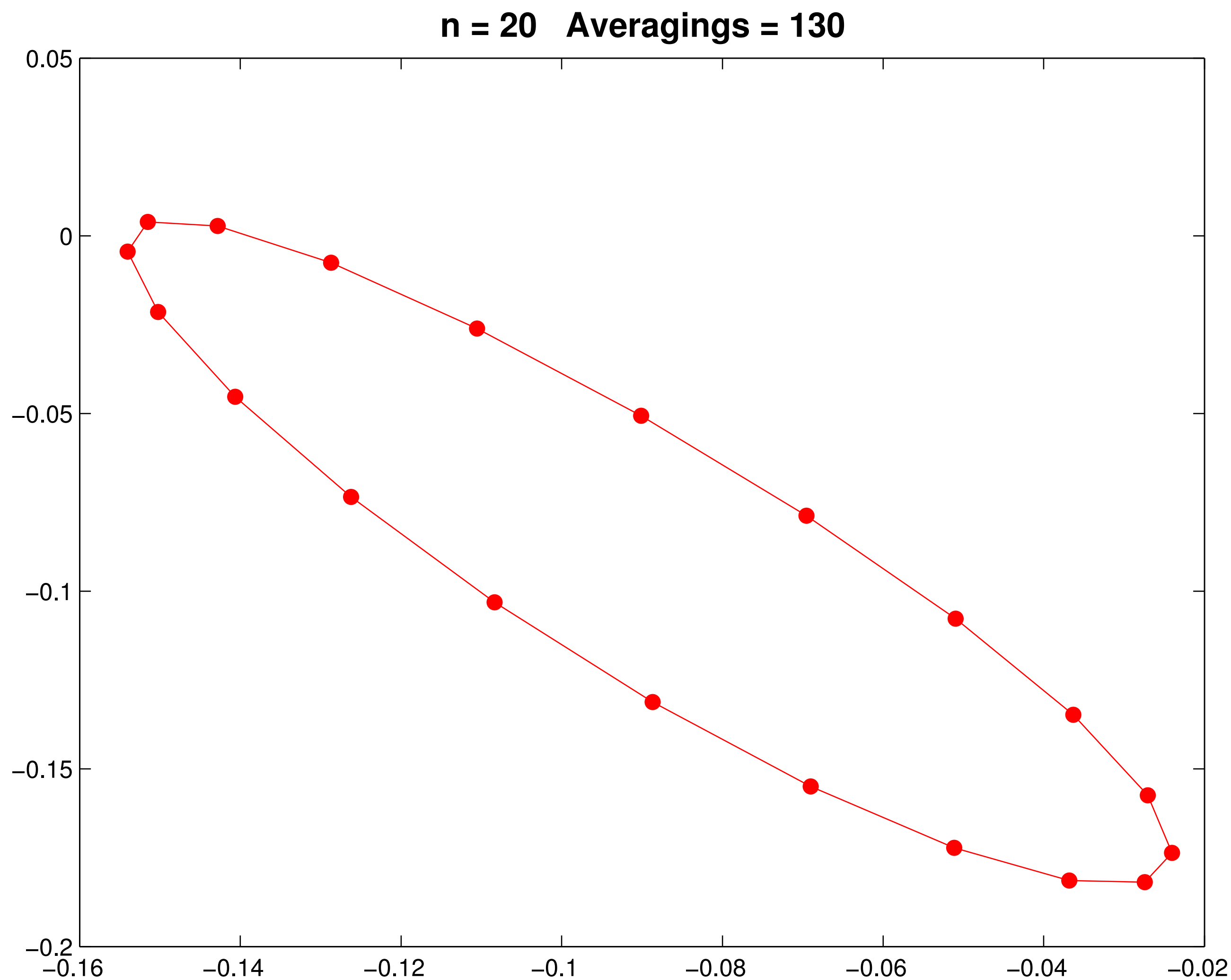
n = 20   Averagings = 65
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.25
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1

Solution 2



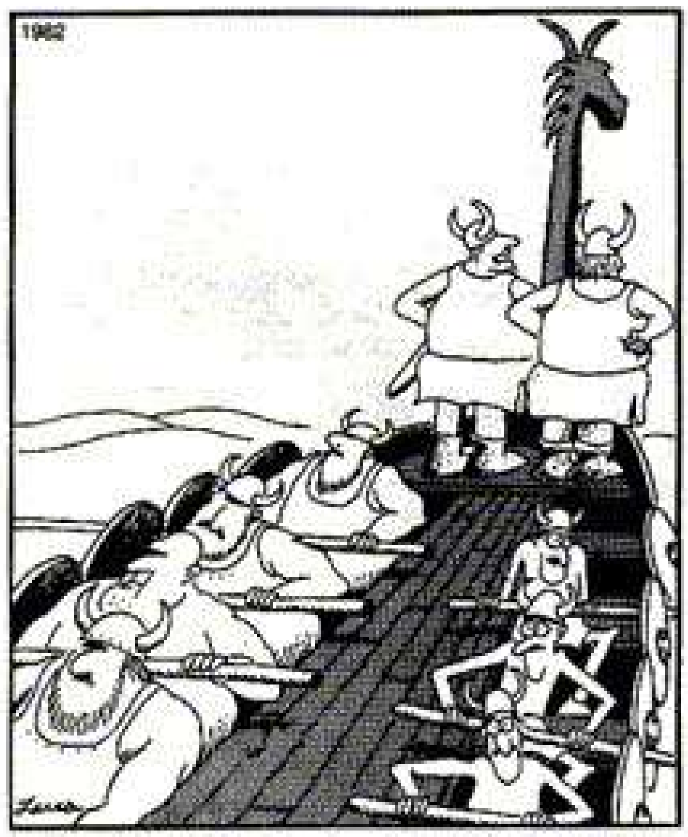
n = 20 Averagings = 65
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.25
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1

Solution 2



n = 20 Averagings = 130
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.16
−0.14
−0.12
−0.1
−0.08
−0.06
−0.04
−0.02

The Scandinavian Analysis



"I've got it, too, Omar.... a strange feeling like we've just been going in circles."

"...a strange feeling like we have just been going in circles."

A Senior-Level Scientific Computing Problem

Given a polygon with centroid (0,0), generate a a new polygon by connecting the midpoints of its sides. The x and y vertex vectors should always have unit 2-norm.

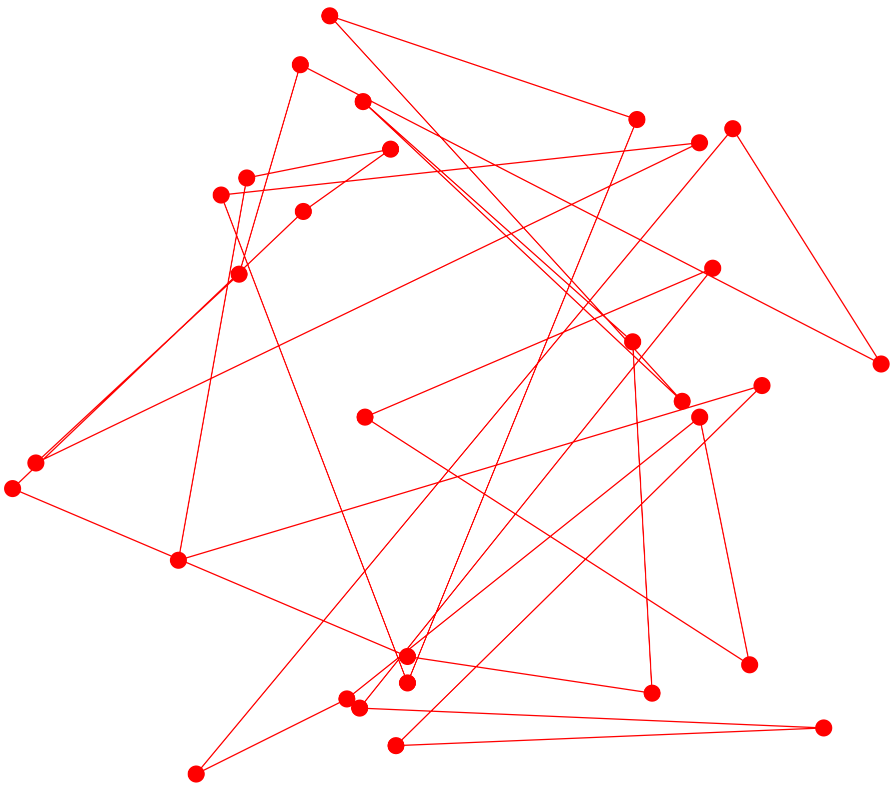
Repeat the process many times, show what happens graphically, and EXPLAIN WHAT YOU SEE.

The Iteration

```
% Random vectors with mean zero...

x = rand(n,1); x = x - mean(x); x = x/norm(x);
y = rand(n,1); y = y - mean(y); y = y/norm(y);

for i=1:nRepeat

   % Connect midpoints, scale, and plot...
   x = (x + [x(2:n);x(1)])/2;   x = x/norm(x);
   y = (y + [y(2:n);y(1)])/2;   y = y/norm(y);
   plot(x,y)

end
```

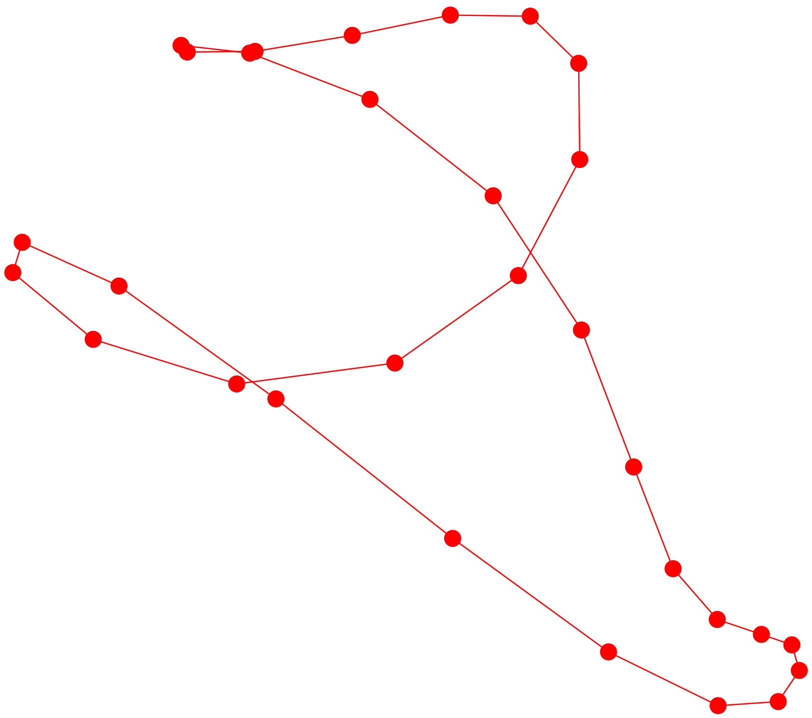
Solution 3

A Random Polygon (n = 30)



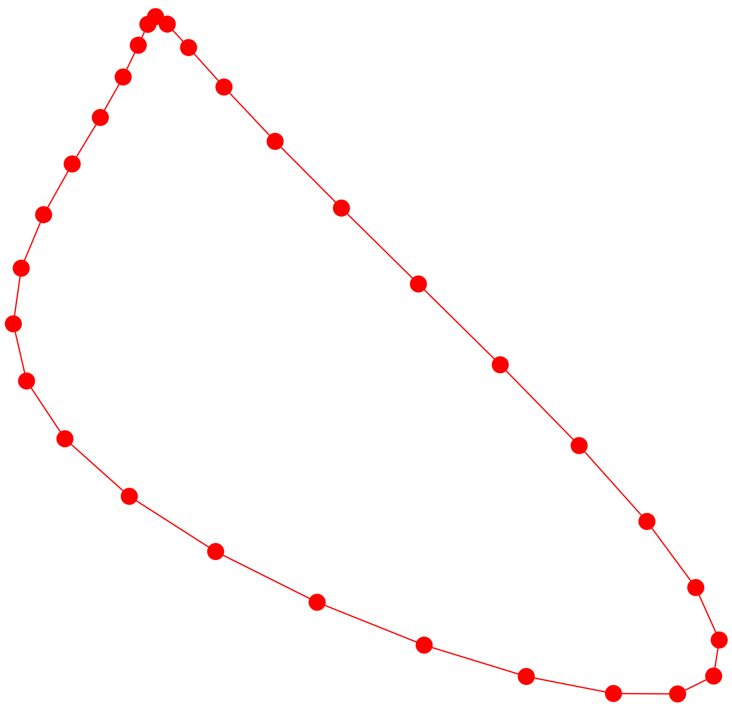
Solution 3

n = 30   Averagings = 20



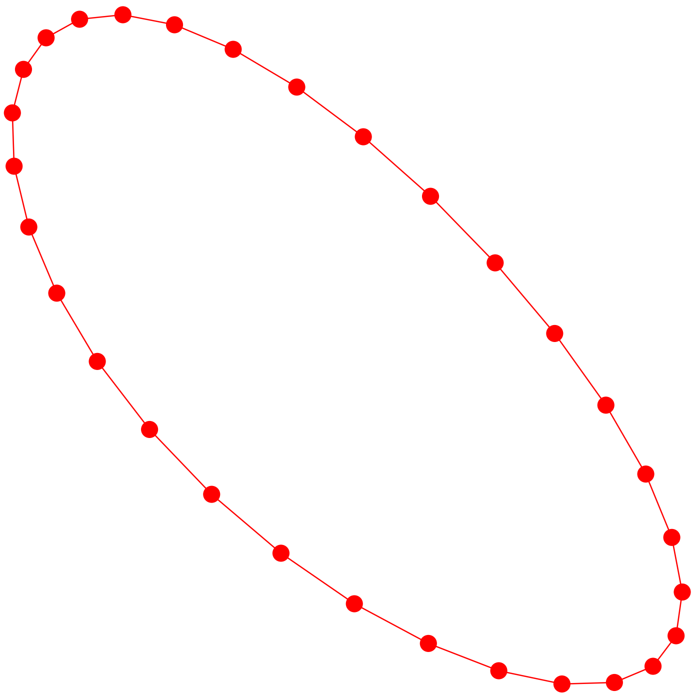
Solution 3

n = 30   Averagings = 100



Solution 3

n = 30   Averagings = 225



Matrix Description

Midpoint generation is a matrix-vector product computation...

$$\begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \\ \widehat{x}_5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \\ x_4 + x_5 \\ x_5 + x_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\Uparrow$$

$$M_5$$

$$\Downarrow$$

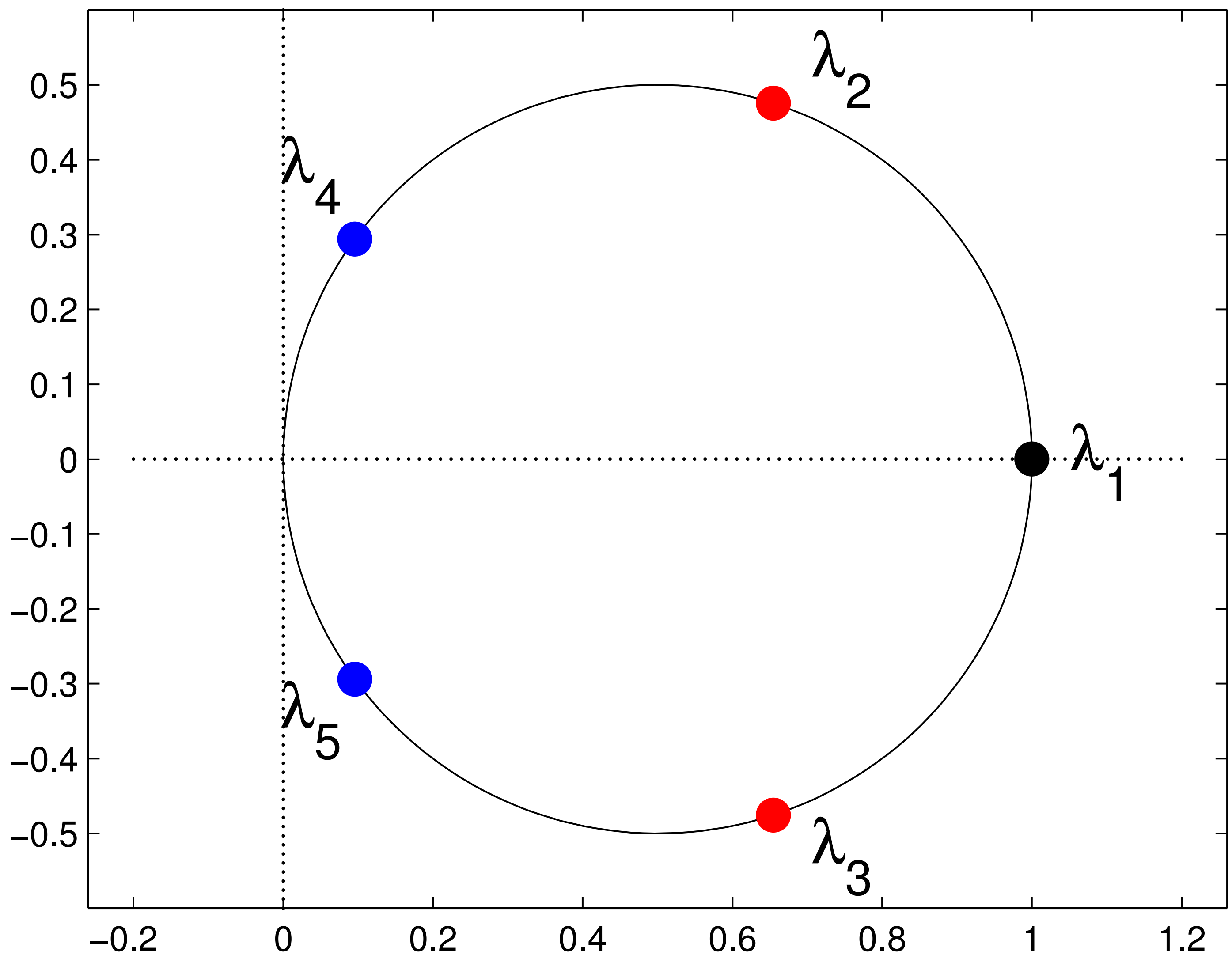
$$\begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \widehat{y}_3 \\ \widehat{y}_4 \\ \widehat{y}_5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} y_1 + y_2 \\ y_2 + y_3 \\ y_3 + y_4 \\ y_4 + y_5 \\ y_5 + y_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

The Iteration

```
% Random vectors with mean zero...
x = rand(n,1); x = x - mean(x); x = x/norm(x);
y = rand(n,1); y = y - mean(y); y = y/norm(y);
for i=1:nRepeat
   % Connect midpoints, scale, and plot...
   x = M(n)*x;   x = x/norm(x);
   y = M(n)*y;   y = y/norm(y);
   plot(x,y)
end
```

Side-by-side instances of the power method.

The Eigenvalues of M_5



The Eigenspaces of M_5

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} \cos(0) \\ \cos(2\pi/5) \\ \cos(4\pi/5) \\ \cos(6\pi/5) \\ \cos(8\pi/5) \end{bmatrix}, \begin{bmatrix} \sin(0) \\ \sin(2\pi/5) \\ \sin(4\pi/5) \\ \sin(6\pi/5) \\ \sin(8\pi/5) \end{bmatrix} \right\}, \left\{ \begin{bmatrix} \cos(0) \\ \cos(4\pi/5) \\ \cos(8\pi/5) \\ \cos(12\pi/5) \\ \cos(16\pi/5) \end{bmatrix}, \begin{bmatrix} \sin(0) \\ \sin(4\pi/5) \\ \sin(8\pi/5) \\ \sin(12\pi/5) \\ \sin(16\pi/5) \end{bmatrix} \right\}$$

$$\lambda_1 = 1 \qquad |\lambda_2| = |\lambda_2| = \cos^2\left(\frac{\pi}{5}\right) \qquad |\lambda_4| = |\lambda_5| = \cos^2\left(\frac{2\pi}{5}\right)$$

The Damping Factor

$$\frac{|\lambda_4|}{|\lambda_2|} \quad = \quad \frac{\cos\left(\frac{2\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)} \quad = \quad 1 \; - \; \frac{3}{2}\left(\frac{\pi}{n}\right)^2 \; + \; O\left(\frac{1}{n^4}\right)$$

After k averagings:

$$\text{dist(Vertices , Limiting Ellipse)} \quad \approx \quad \left(\frac{|\lambda_4|}{|\lambda_2|}\right)^k$$

At the Start...

$$x^{(0)} = \alpha_1 \begin{bmatrix} \cos\left(\frac{0\pi}{5}\right) \\ \cos\left(\frac{2\pi}{5}\right) \\ \cos\left(\frac{4\pi}{5}\right) \\ \cos\left(\frac{6\pi}{5}\right) \\ \cos\left(\frac{8\pi}{5}\right) \end{bmatrix} + \alpha_2 \begin{bmatrix} \sin\left(\frac{0\pi}{5}\right) \\ \sin\left(\frac{2\pi}{5}\right) \\ \sin\left(\frac{4\pi}{5}\right) \\ \sin\left(\frac{6\pi}{5}\right) \\ \sin\left(\frac{8\pi}{5}\right) \end{bmatrix} + \alpha_3 \begin{bmatrix} \cos\left(\frac{0\pi}{5}\right) \\ \cos\left(\frac{4\pi}{5}\right) \\ \cos\left(\frac{8\pi}{5}\right) \\ \cos\left(\frac{12\pi}{5}\right) \\ \cos\left(\frac{16\pi}{5}\right) \end{bmatrix} + \alpha_4 \begin{bmatrix} \sin\left(\frac{0\pi}{5}\right) \\ \sin\left(\frac{4\pi}{5}\right) \\ \sin\left(\frac{8\pi}{5}\right) \\ \sin\left(\frac{12\pi}{5}\right) \\ \sin\left(\frac{16\pi}{5}\right) \end{bmatrix}$$

$$y^{(0)} = \beta_1 \begin{bmatrix} \text{ditto} \end{bmatrix} + \beta_2 \begin{bmatrix} \text{ditto} \end{bmatrix} + \beta_3 \begin{bmatrix} \text{ditto} \end{bmatrix} + \beta_4 \begin{bmatrix} \text{ditto} \end{bmatrix}$$

In the Limit...

After k averagings, vertex vectors $x^{(k)}$ and $y^{(k)}$ are unit vectors in the direction of

$$M_5^k x^{(0)} \approx \alpha_1 M_5^k \begin{bmatrix} \cos\left(\frac{0\pi}{5}\right) \\ \cos\left(\frac{2\pi}{5}\right) \\ \cos\left(\frac{4\pi}{5}\right) \\ \cos\left(\frac{6\pi}{5}\right) \\ \cos\left(\frac{8\pi}{5}\right) \end{bmatrix} + \alpha_2 M_5^k \begin{bmatrix} \sin\left(\frac{0\pi}{5}\right) \\ \sin\left(\frac{2\pi}{5}\right) \\ \sin\left(\frac{4\pi}{5}\right) \\ \sin\left(\frac{6\pi}{5}\right) \\ \sin\left(\frac{8\pi}{5}\right) \end{bmatrix}$$

$$M_5^k y^{(0)} \approx \beta_1 M_5^k \left[\ \text{ditto}\ \right] + \beta_2 M_5^k \left[\ \text{ditto}\ \right]$$

What Those Unit Vectors Are.....

$$x^{(k)} \approx \cos(\theta_{\mathbf{x}}) \begin{bmatrix} \cos\left(\frac{(0+k)\pi}{5}\right) \\ \cos\left(\frac{(2+k)\pi}{5}\right) \\ \cos\left(\frac{(4+k)\pi}{5}\right) \\ \cos\left(\frac{(6+k)\pi}{5}\right) \\ \cos\left(\frac{(8+k)\pi}{5}\right) \end{bmatrix} + \sin(\theta_{\mathbf{x}}) \begin{bmatrix} \sin\left(\frac{(0+k)\pi}{5}\right) \\ \sin\left(\frac{(2+k)\pi}{5}\right) \\ \sin\left(\frac{(4+k)\pi}{5}\right) \\ \sin\left(\frac{(6+k)\pi}{5}\right) \\ \sin\left(\frac{(8+k)\pi}{5}\right) \end{bmatrix}$$

$$y^{(k)} \approx \cos(\theta_{\mathbf{y}}) \begin{bmatrix} \text{ditto} \end{bmatrix} + \sin(\theta_{\mathbf{y}}) \begin{bmatrix} \text{ditto} \end{bmatrix}$$

The Four Magic Numbers.....

$$\cos(\theta_x) \;=\; \frac{c^T x^{(0)}}{\sqrt{(c^T x^{(0)})^2 \;+\; (s^T x^{(0)})^2}}$$

$$c^T = \left[\; \cos\left(\frac{0\pi}{5}\right) \quad \cos\left(\frac{2\pi}{5}\right) \quad \cos\left(\frac{4\pi}{5}\right) \quad \cos\left(\frac{6\pi}{5}\right) \quad \cos\left(\frac{8\pi}{5}\right) \;\right]$$

$$s^T = \left[\; \sin\left(\frac{0\pi}{5}\right) \quad \sin\left(\frac{2\pi}{5}\right) \quad \sin\left(\frac{4\pi}{5}\right) \quad \sin\left(\frac{6\pi}{5}\right) \quad \sin\left(\frac{8\pi}{5}\right) \;\right]$$

The recipes for $\sin(\theta_x)$, $\cos(\theta_y)$, and $\sin(\theta_y)$ are similar.

Where the Vertices Sit.....

$$x(t) \approx \cos(\theta_{\mathbf{x}})\cos(t) + \sin(\theta_{\mathbf{x}})\sin(t)$$

$$y(t) \approx \cos(\theta_{\mathbf{y}})\cos(t) + \sin(\theta_{\mathbf{y}})\sin(t)$$

This describes an ellipse!

What Are Its Semiaxes and Tilt?

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx \begin{bmatrix} \cos(\theta_{\mathbf{x}}) & \sin(\theta_{\mathbf{x}}) \\ \cos(\theta_{\mathbf{y}}) & \sin(\theta_{\mathbf{y}}) \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$= U \begin{bmatrix} \sigma_{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & \sigma_{\mathbf{2}} \end{bmatrix} V^T \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

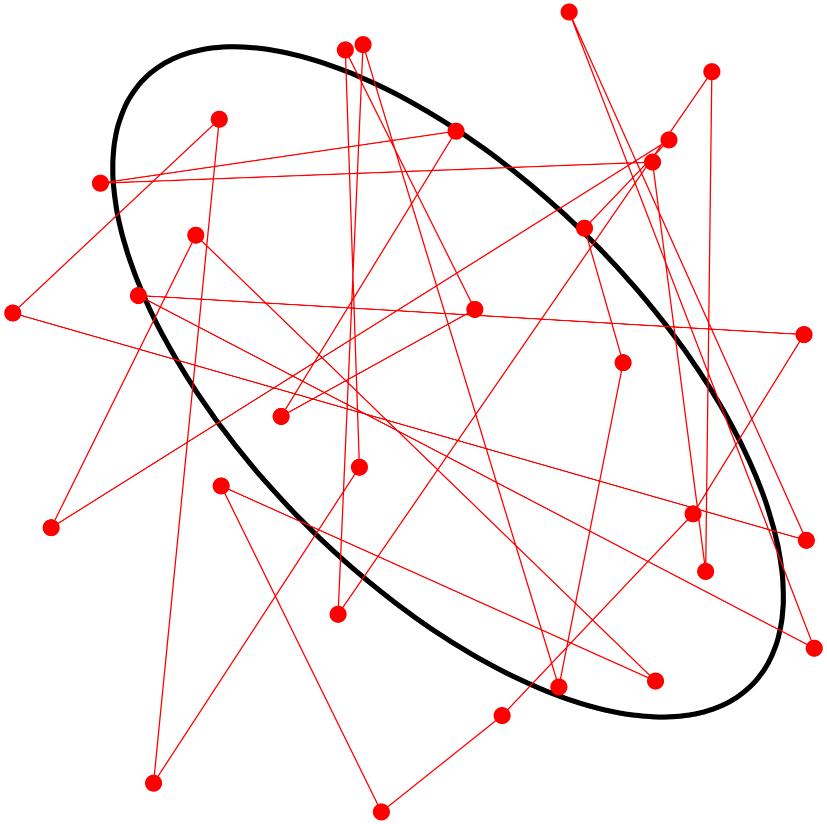
The SVD Tells All...

The Limiting Ellipse

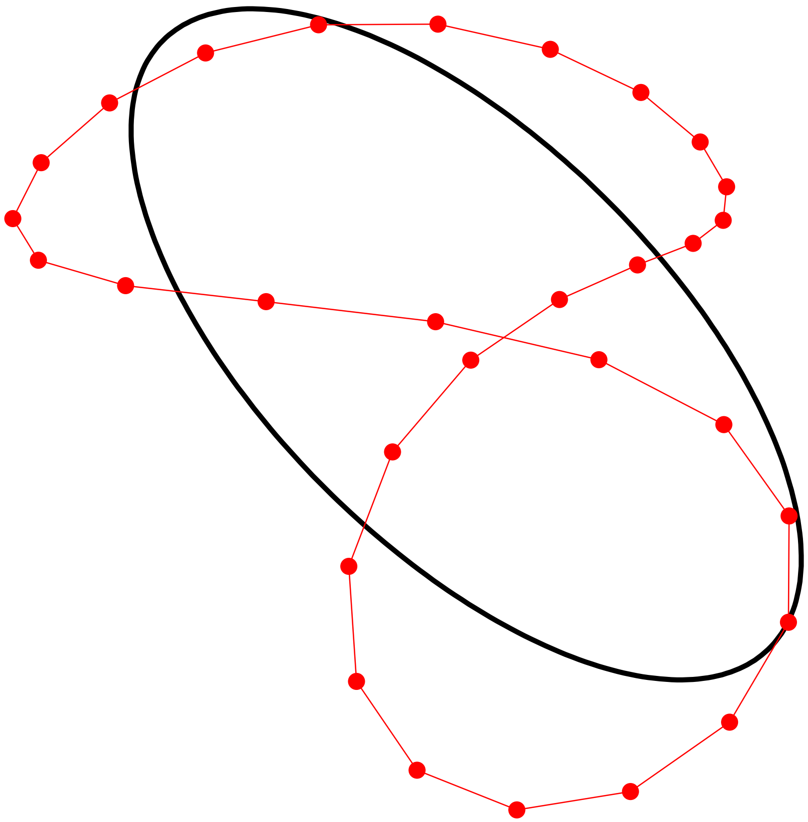
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx \begin{bmatrix} \cos(\mathbf{45}) & -\sin(\mathbf{45}) \\ \cos(\mathbf{45}) & \sin(\mathbf{45}) \end{bmatrix} \begin{bmatrix} \sigma_{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & \sigma_{\mathbf{2}} \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$\sigma_1 = \frac{2}{\sqrt{n}} \cdot \cos\left(\frac{\theta_y - \theta_x}{2}\right) \qquad \sigma_2 = \frac{2}{\sqrt{n}} \cdot \sin\left(\frac{\theta_y - \theta_x}{2}\right)$$

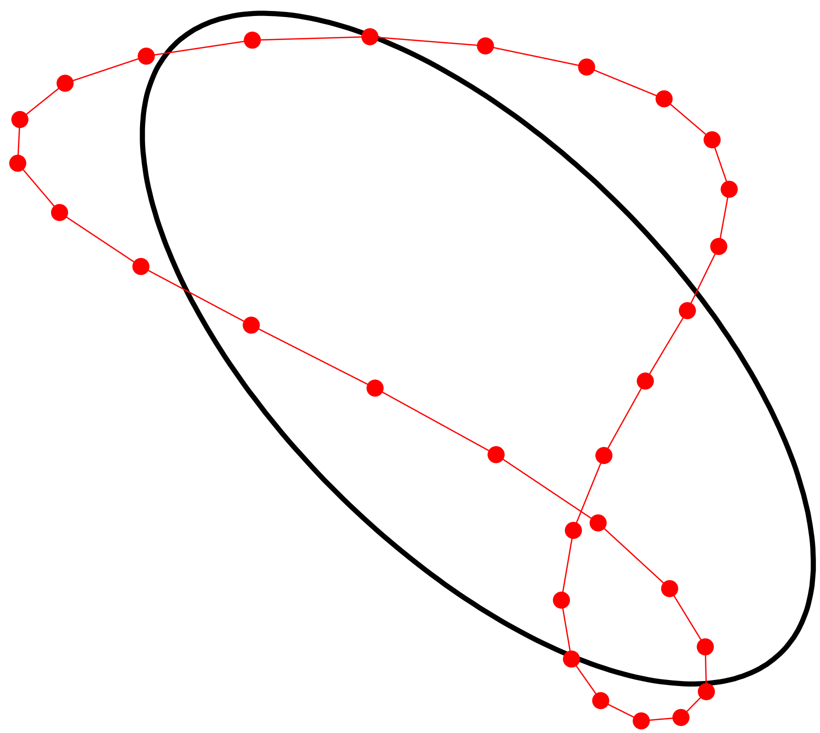
An Initial "Random" Polygon



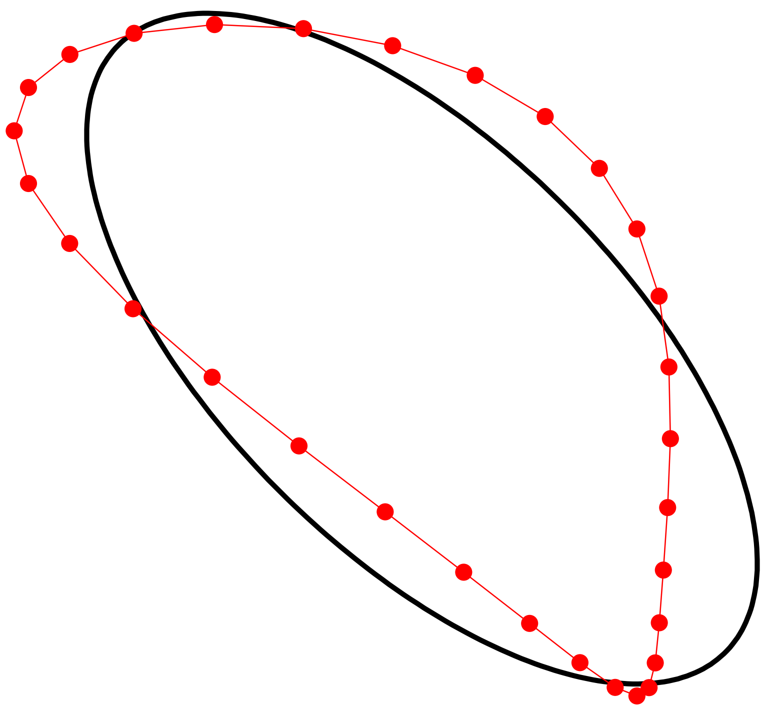
After 32 Averagings...



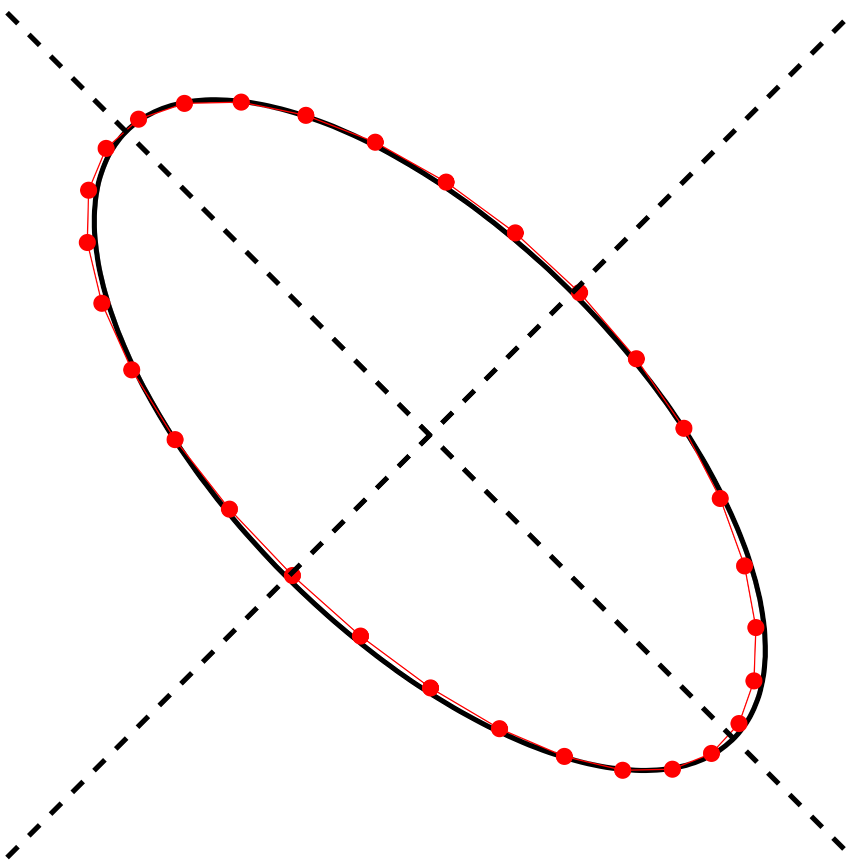
After 80 Averagings...



After 140 Averagings...

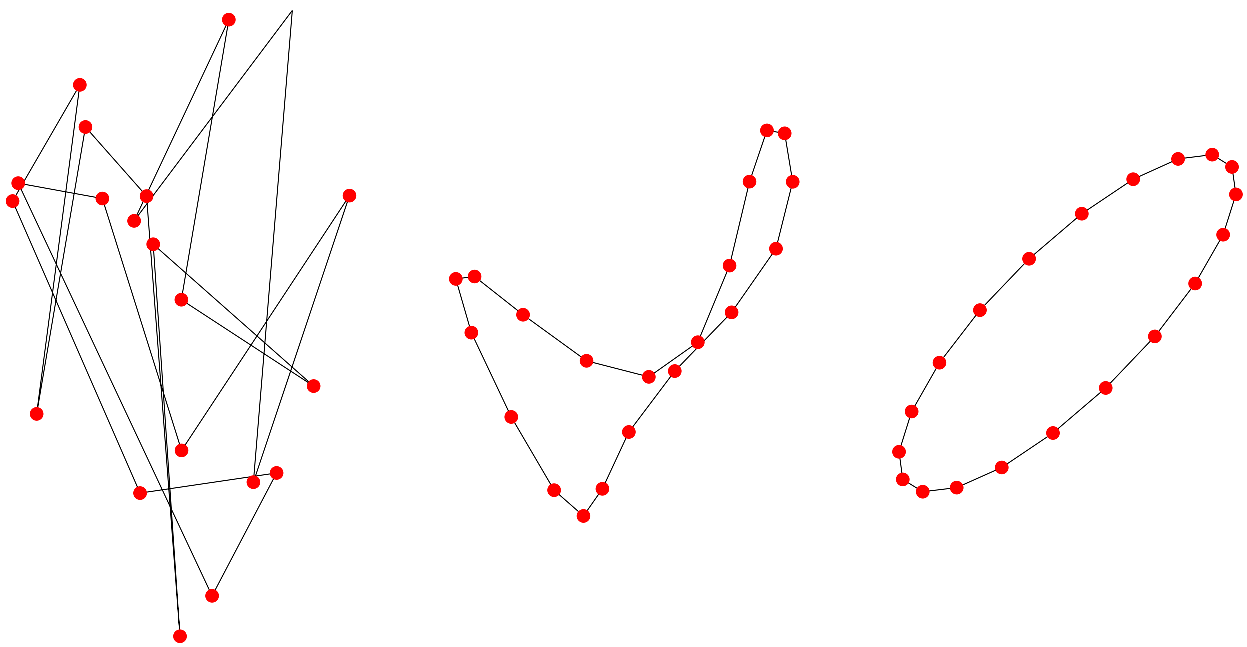


After 278 Averagings...



An Undergraduate Research Problem

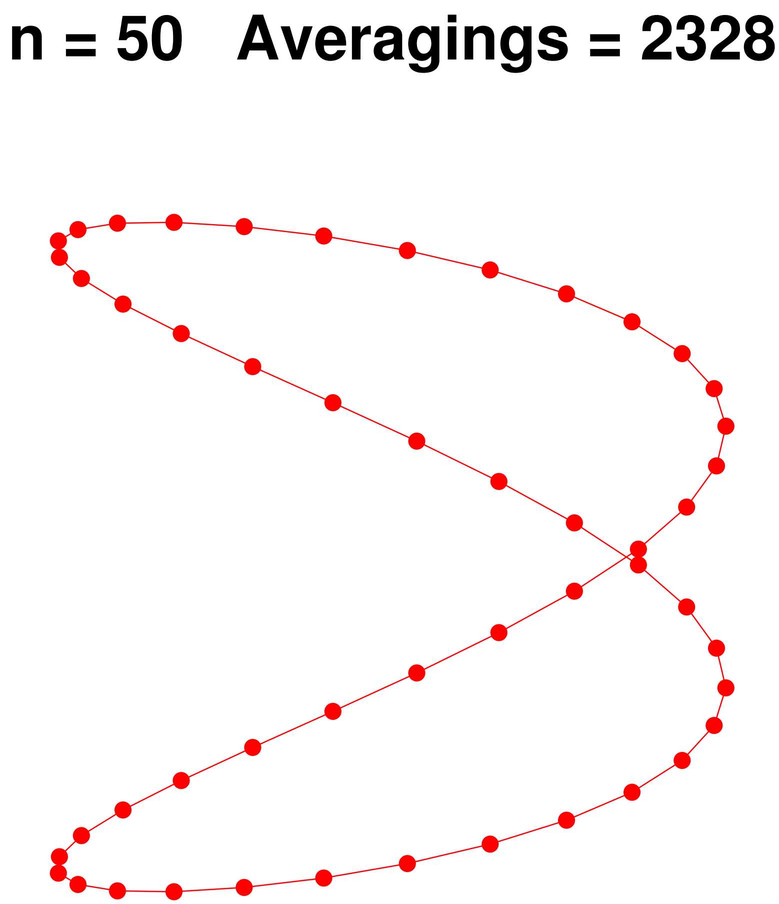
Explain why the limiting ellipse has a 45-degree tilt? What are the lengths of its semiaxes?



A. Elmachtaub and C. Van Loan (2010). "From Random Polygon to Ellipse," *SIAM Review*, 52, 151-170.

When the Initial Vertex Vectors Are Deficient...

x orthoginal to D2



More Interesting Behavior...

1. What happens if we scale by other norms? E.g.,

$$x = x / \| \, x \, \|_p$$

2. What happens if we use alternative “midpoints”? E.g.,

$$x_i = \lambda \cdot x_i \; + \; (1 - \lambda) \cdot x_{i+1}$$

The Divitz Metaphor Analysis...

Who would want to go out

with somebody who averages polygons?

The Problem is a Metaphor for Computational Science and Engineering

First:

We *experimented* with a simple iteration and observed that it transforms something that is chaotic and rough into something that is organized and smooth.

The Problem is a Metaphor for Computational Science and Engineering

Second:

As a step towards explaining the limiting behavior of the polygon sequence, we described the averaging process using *matrix-vector notation*.

The Problem is a Metaphor for Computational Science and Engineering

Third:

This led to an *eigenanalysis*, the identification of a crucial *invariant subspace*, and a vertex-vector *convergence analysis*.

The Problem is a Metaphor for Computational Science and Engineering

Fourth:

We then used the *singular value decomposition* to connect our algebraic manipulations to a simple underlying geometry.

High-Level Summary

A rowboat is a gondola...

Voyage of Life: Childhood

Lucky to be here.

Voyage of Life: Youth

The view beats the boat. Any boat.

Voyage of Life: Middle Age

Research w/o Rudder? Who cares because...

Voyage of Life: Youth

You can always start over.