Bridging the Gap from Matrix to Tensor

Block Tensor Computations

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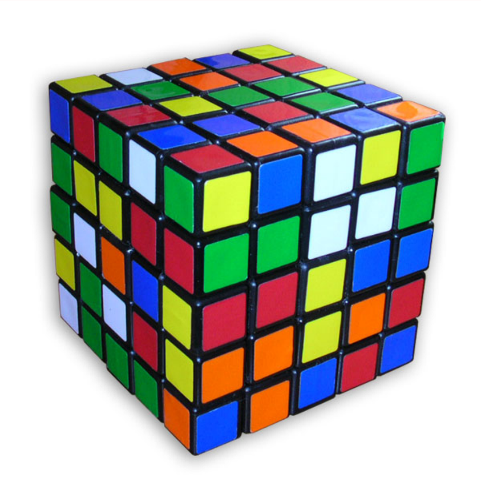
A 5-by-5-by-5 Block Tensor...



Assuming that each of the 125 blocks is an order-3 tensor and that they all they "fit together."

More Generally, What is a Block Tensor?

Informal Definition

A tensor whose entries are other tensors of the same order.

Example

If $\mathcal{A} \in \mathbb{R}^{9\times 5\times 8\times 7}$ and

$$\begin{aligned} 1{:}9 &= \left[\; 1{:}2 \;\middle|\; 3{:}5 \;\middle|\; 6{:}9 \;\right] \\ 1{:}5 &= \left[\; 1{:}3 \;\middle|\; 4{:}5 \;\right] \\ 1{:}8 &= \left[\; 1{:}2 \;\middle|\; 3{:}4 \;\middle|\; 5{:}6 \;\middle|\; 7{:}8 \;\right] \\ 1{:}7 &= \left[\; 1{:}4 \;\middle|\; 5{:}7 \;\right] \end{aligned}$$

then $\mathcal{A}$ can be regarded as a 3-by-2-by-4-by-2 block tensor.

The (2,1,3,2) block: $\mathcal{A}_{2132} = \mathcal{A}(3{:}5, 1{:}3, 5{:}6, 5{:}7)$.

The Context

The Next BIG Thing?

Scalar-Level Thinking

1960's $\Downarrow$ $\Leftarrow$ The factorization paradigm: LU, LDL^T, QR, $U\Sigma V^T$, etc.

Matrix-Level Thinking

1980's $\Downarrow$ $\Leftarrow$ Cache utilization, parallel computing, LAPACK, etc.

Block Matrix-Level Thinking

2000's $\Downarrow$ $\Leftarrow$ New applications, factorizations, data structures, nonlinear analysis, optimization strategies, etc.

Tensor-Level Thinking

The Context (Continued)

The Changing Definition of "Big"

In **Matrix Computations**, to say that $A \in \mathbb{R}^{n_1 \times n_2}$ is "big" is to say that both n_1 and n_2 are big.

In **Tensor Computations**, to say that $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ is "big" is to say that $n_1 n_2 \cdots n_d$ is big and this need not require big n_k. E.g. $n_1 = n_2 = \cdots = n_{1000} = 2$.

Algorithms that beat the "curse of dimensionality" will induce a transition...

Matrix-Based Scientific Computation

$\Downarrow$

Tensor-Based Scientific Computation

The Context (Continued)

Tensor-Related Presentations at Householder XVIII

Karen Braman	Tamara Kolda
Lars Eldén	Lek-Heng Lim
Shmuel Friedland	Ivan Oseledets
Thomas Huckle	Stefan Ragnarsson
Misha Kilmer	Berkant Savas

Sabine Van Huffel

The Fringe Benefits of Blocking In Matrix Computations

Insight

$$\text{FFT:} \quad F_{2m}x \;=\; \left[\begin{array}{cc} I_m & \text{diag}(\omega_n^k) \\ I_m & -\text{diag}(\omega_n^k) \end{array}\right] \left[\begin{array}{cc} F_m & 0 \\ 0 & F_m \end{array}\right] \left[\begin{array}{c} x(2{:}2{:}2m) \\ x(1{:}2{:}2m) \end{array}\right]$$

Data Re-Use

Level-3 BLAS $\Longrightarrow$ Block LU, QR, etc

Generalization

Lanczos $\Longrightarrow$ Block Lanczos

Givens Rotations $\Longrightarrow$ CS Decomposition

Versions of these stories are beginning to play out in tensor computations and this talk is about that.

Acknowledgments

Lieven De Lathauwer

PhD Thesis (1997)

Recent SIMAX papers that connect the Parafac and Tucker representations through a block-structured core tensor. (2010)

What follows is based upon ongoing research and two papers...

Block Tensors and Symmetric Embeddings (with Stefan Ragnarsson)

Block Tensor Unfoldings (with Stefan Ragnarsson)

NSF DMS-1016284

Tensor Unfolding

A Common Framework for Tensor Computations...

1. Reshape **tensor** $\mathcal{A}$ into a **matrix** A.

2. Through matrix computations, discover things about **matrix** A.

3. Draw conclusions about **tensor** $\mathcal{A}$ based on what is learned about **matrix** A.

"Reshape" $\equiv$ **"Unfold"** $\equiv$ **"Matricize"** $\equiv$ **"Flatten"**

Modal Unfoldings

A Mode-1 Unfolding of $\mathcal{A} \in \mathbb{R}^{4\times 3\times 2}$

$$\mathcal{A}_{(1)} = \begin{bmatrix} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} \\ a_{411} & a_{421} & a_{431} & a_{412} & a_{422} & a_{432} \end{bmatrix}$$

(1,1) (2,1) (3,1) (1,2) (2,2) (3,2)

A Mode-2 Unfolding of $\mathcal{A} \in \mathbb{R}^{4\times 3\times 2}$

$$\mathcal{A}_{(2)} = \begin{bmatrix} a_{111} & a_{211} & a_{311} & a_{411} & a_{112} & a_{212} & a_{312} & a_{412} \\ a_{121} & a_{221} & a_{321} & a_{421} & a_{122} & a_{222} & a_{322} & a_{422} \\ a_{131} & a_{231} & a_{331} & a_{431} & a_{132} & a_{232} & a_{332} & a_{432} \end{bmatrix}$$

(1,1) (2,1) (3,1) (4,1) (1,2) (2,2) (3,2) (4,2)

More General Unfoldings

If $\mathcal{A} \in \mathbb{R}^{2\times 3\times 2\times 2\times 3}$, $r = [1,\ 2,\ 4]$, and $c = [3,\ 5]$, then

$$\mathcal{A}_{r\times c} = \begin{array}{c} \\ \left[\begin{array}{cccccc} (1,1) & (2,1) & (1,2) & (2,2) & (1,3) & (2,3) \\ a_{11111} & a_{11211} & a_{11112} & a_{11212} & a_{11113} & a_{11213} \\ a_{21111} & a_{21211} & a_{21112} & a_{21212} & a_{21113} & a_{21213} \\ a_{12111} & a_{12211} & a_{12112} & a_{12212} & a_{12113} & a_{12213} \\ a_{22111} & a_{22211} & a_{22112} & a_{22212} & a_{22113} & a_{22213} \\ a_{13111} & a_{13211} & a_{13112} & a_{13212} & a_{13113} & a_{13213} \\ a_{23111} & a_{23211} & a_{23112} & a_{23212} & a_{23113} & a_{23213} \\ a_{11121} & a_{11221} & a_{11122} & a_{11222} & a_{11123} & a_{11223} \\ a_{21121} & a_{21221} & a_{21122} & a_{21222} & a_{21123} & a_{21223} \\ a_{12121} & a_{12221} & a_{12122} & a_{12222} & a_{12123} & a_{12223} \\ a_{22121} & a_{22221} & a_{22122} & a_{22222} & a_{22123} & a_{22223} \\ a_{13121} & a_{13221} & a_{13122} & a_{13222} & a_{13123} & a_{13223} \\ a_{23121} & a_{23221} & a_{23122} & a_{23222} & a_{23123} & a_{23223} \end{array}\right] \end{array} \begin{array}{c} \\ (1,1,1) \\ (2,1,1) \\ (1,2,1) \\ (2,2,1) \\ (1,3,1) \\ (2,3,1) \\ (1,1,2) \\ (2,1,2) \\ (1,2,2) \\ (2,2,2) \\ (1,3,2) \\ (2,3,2) \end{array}$$

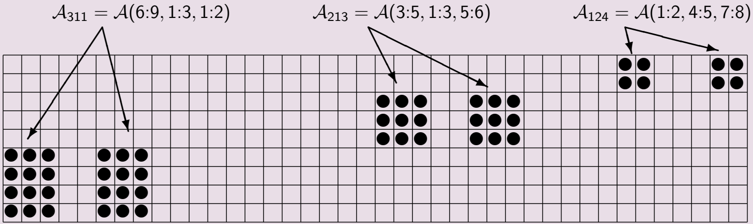
With the Tensor Toolbox..

```
A = tenrand([2 3 2 2 3]);

r = [ 1 2 4];

c = [3 5];

Amat = tenmat(A,r,c);
```

Kolda and Bader (2006)

Block Unfoldings

Tensor Blocks Are Not Contiguous in a Vec-Based Unfolding



Tensor Blocks Are Contiguous in a Block Unfolding

2	$(\mathcal{A}_{111})_{(1)}$	$(\mathcal{A}_{121})_{(1)}$	$(\mathcal{A}_{112})_{(1)}$	$(\mathcal{A}_{122})_{(1)}$	$(\mathcal{A}_{113})_{(1)}$	$(\mathcal{A}_{123})_{(1)}$	$(\mathcal{A}_{114})_{(1)}$	$(\mathcal{A}_{124})_{(1)}$
3	$(\mathcal{A}_{211})_{(1)}$	$(\mathcal{A}_{221})_{(1)}$	$(\mathcal{A}_{212})_{(1)}$	$(\mathcal{A}_{222})_{(1)}$	$(\mathcal{A}_{213})_{(1)}$	$(\mathcal{A}_{223})_{(1)}$	$(\mathcal{A}_{214})_{(1)}$	$(\mathcal{A}_{224})_{(1)}$
4	$(\mathcal{A}_{311})_{(1)}$	$(\mathcal{A}_{321})_{(1)}$	$(\mathcal{A}_{312})_{(1)}$	$(\mathcal{A}_{322})_{(1)}$	$(\mathcal{A}_{313})_{(1)}$	$(\mathcal{A}_{323})_{(1)}$	$(\mathcal{A}_{314})_{(1)}$	$(\mathcal{A}_{324})_{(1)}$
	6	4	6	4	6	4	6	4

Vec vs BlockVec

The Vec Ordering...

A =

1	12	23	34	45	56	67	78	89
2	13	24	35	46	57	68	79	90
3	14	25	36	47	58	69	80	91
4	15	26	37	48	59	70	81	92
5	16	27	38	49	60	71	82	93
6	17	28	39	50	61	72	83	94
7	18	29	40	51	62	73	84	95
8	19	30	41	52	63	74	85	96
9	20	31	42	53	64	75	86	97
10	21	32	43	54	65	76	87	98
11	22	33	44	55	66	77	88	99

If `v = Vec(A)` then `v(58) = A(3,6).`

The BlockVec Ordering...

A =

1	3	23	25	27	29	67	69	71
2	4	24	26	28	30	68	70	72
5	10	31	36	41	46	73	78	83
6	11	32	37	42	47	74	79	84
7	12	33	38	43	48	75	80	85
8	13	34	39	44	49	76	81	86
9	14	35	40	45	50	77	82	87
15	19	51	55	59	63	88	92	96
16	20	52	56	60	64	89	93	97
17	21	53	57	61	65	90	94	98
18	22	54	58	62	66	91	95	99

If `v = BlockVec(A)` then `v(46) = A(3,6)`.

Vec vs BlockVec

Vec for Tensors...

If $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, then

$$\text{vec}\,(\mathcal{A}) \;=\; \begin{bmatrix} \text{vec}(\mathcal{A}(:,:,1)) \\ \vdots \\ \text{vec}(\mathcal{A}(:,:,n_3)) \end{bmatrix}$$

BlockVec for Tensors

If $\mathcal{A}$ is a 2-by-2-by-2 block tensor, then

$$\text{BlockVec}(\mathcal{A}) \;=\; \begin{bmatrix} \text{vec}(\mathcal{A}_{111}) \\ \text{vec}(\mathcal{A}_{211}) \\ \text{vec}(\mathcal{A}_{121}) \\ \text{vec}(\mathcal{A}_{221}) \\ \text{vec}(\mathcal{A}_{112}) \\ \text{vec}(\mathcal{A}_{212}) \\ \text{vec}(\mathcal{A}_{122}) \\ \text{vec}(\mathcal{A}_{222}) \end{bmatrix}$$

Connections

Basic Theorem

We have a complete specification of the permutation that maps $\text{Vec}(\mathcal{A})$ to $\text{BlockVec}(\mathcal{A})$ where $\mathcal{A}$ is a tensor with blocking M:

$$\text{BlockVec}(\mathcal{A}) \;=\; P_M \,\text{Vec}(\mathcal{A})$$

It is an intricate combination of perfect shuffles.

Block Unfolding Theorem

Relates the $r \times c$ vec-based unfolding to the corresponding block unfolding:

$$\mathcal{A}_{R\times C} \;=\; P_R \,\mathcal{A}_{r\times c}\, P_C^T$$

Blocks in the tensor become contiguous blocks in the unfolding.

Some ramifications...

Symmetric Embedding and Tensor Rank

Is There a Tensor Analog of This?

The "Sym" of a Matrix

$$\text{sym}(A) \; = \; \left[\begin{array}{cc} 0 & A \\ A^T & 0 \end{array} \right] \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$$

The SVD of A Relates to the EVD of sym(A)

If $A = U \cdot \text{diag}(\sigma_i) \cdot V^T$ is the SVD of $A \in \mathbb{R}^{n_1 \times n_2}$, then for $k = 1{:}\text{rank}(A)$

$$\left[\begin{array}{cc} 0 & A \\ A^T & 0 \end{array} \right] \left[\begin{array}{c} u_k \\ \pm v_k \end{array} \right] \; = \; \pm\sigma_k \left[\begin{array}{c} u_k \\ \pm v_k \end{array} \right]$$

where $u_k = U(:,k)$ and $v_k = V(:,k)$.

Try to shed light on the tensor rank problem and connect some well-known power iterations.

Tensor Transposition: The Order-3 Case

Six possibilities...

If $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, then there are $6 = 3!$ possible transpositions identified by the notation $\mathcal{A}^{<[i\ j\ k]>}$ where $[i\ j\ k]$ is a permutation of [1 2 3]:

$$\mathcal{B} = \left\{ \begin{array}{c} \mathcal{A}^{<[1\ 2\ 3]>} \\ \mathcal{A}^{<[1\ 3\ 2]>} \\ \mathcal{A}^{<[2\ 1\ 3]>} \\ \mathcal{A}^{<[2\ 3\ 1]>} \\ \mathcal{A}^{<[3\ 1\ 2]>} \\ \mathcal{A}^{<[3\ 2\ 1]>} \end{array} \right\} \quad \Longrightarrow \quad \left\{ \begin{array}{c} b_{ijk} \\ b_{ikj} \\ b_{jik} \\ b_{jki} \\ b_{kij} \\ b_{kji} \end{array} \right\} = a_{ijk}$$

for $i = 1{:}n_1,\ j = 1{:}n_2,\ k = 1{:}n_3$.

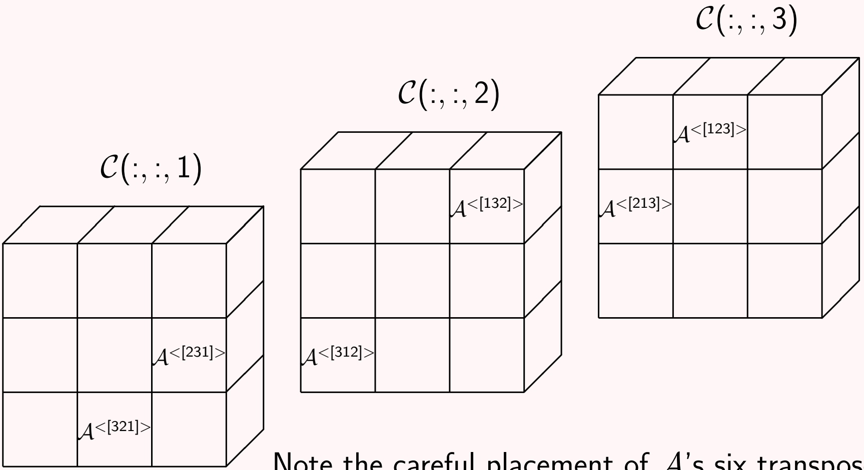
Supersymmetry

Order-3 Definition

$$\mathcal{A}(i,j,k) \;=\; \begin{cases} \mathcal{A}(i,k,j) \\ \mathcal{A}(j,i,k) \\ \mathcal{A}(i,k,i) \\ \mathcal{A}(k,i,j) \\ \mathcal{A}(k,j,i) \end{cases}$$

Symmetric Embedding of a Tensor

An Order-3 Example...



Note the careful placement of $\mathcal{A}$'s six transposes

Rank-1 Tensors

An Example

$$\mathcal{A} = \begin{bmatrix} 10 \\ 20 \\ \color{red}{30} \end{bmatrix} \circ \begin{bmatrix} \color{red}{40} \\ 50 \end{bmatrix} \circ \begin{bmatrix} 60 \\ 70 \\ 80 \\ \color{red}{90} \end{bmatrix} \circ \begin{bmatrix} 100 \\ \color{red}{110} \end{bmatrix}$$

$$\mathcal{A}(3,1,4,2) = \color{red}{30 \cdot 40 \cdot 90 \cdot 110}$$

Some Tensor Rank Definitions (Order-4 Examples)

Outer Product Rank

Shortest sum of the form $\mathcal{A} = \sum_{k=1}^{r} u_k \circ v_k \circ w_k \circ z_k$

Multilinear Rank $[r_1, r_2, r_3, r_4]$

r_1 = rank of the Mode-1 unfolding
r_2 = rank of the Mode-2 unfolding
r_3 = rank of the Mode-3 unfolding
r_4 = rank of the Mode-4 unfolding

Symmetric Rank

Shortest sum of the form $\mathcal{A} = \sum_{k=1}^{r} \alpha_k \cdot u_k \circ u_k \circ u_k \circ u_k$

Some Results for Order-d Tensors

Outer Product Rank

$$d \cdot \text{rank}(\mathcal{A}) \;\leq\; \text{rank}(\text{sym}(\mathcal{A})) \;\leq\; d! \cdot \text{rank}(\mathcal{A})$$

Multilinear Rank

If $\mathcal{A}$ is an order-d tensor with multilinear rank $[r_1, \ldots, r_d]$, then the multilinear rank of sym($\mathcal{A}$) is $[\tilde{r},\ \tilde{r}, \ldots, \tilde{r}]$ where $\tilde{r} = r_1 + \cdots + r_d$.

Contractions and Unfoldings

Vec-Based Contractions and Unfoldings

A Canonical example...

$$\mathcal{C}(i_1, i_2, j_1, j_2, j_3) = \sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2} \mathcal{A}(i_1, i_2, k_1, k_2)\mathcal{B}(k_1, k_2, j_1, j_2, j_3)$$

With Multi-index Notation...

$$\mathcal{C}(\mathbf{i}, \mathbf{j}) = \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{p}} \mathcal{A}(\mathbf{i}, \mathbf{k})\mathcal{B}(\mathbf{k}, \mathbf{j})$$

As a Product of Unfoldings...

$$\mathcal{C}_{[1\ 2]\times[3\ 4\ 5]} = \mathcal{A}_{[1\ 2]\times[3\ 4]} \cdot \mathcal{B}_{[1\ 2]\times[3\ 4\ 5]}$$

A tensor contraction doesn't just look like a matrix multiplication–it IS a matrix multiplication!

Blocked Matrix Multiplication

Visualization

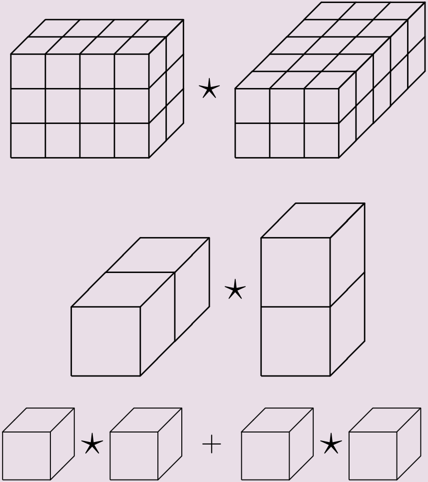
$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix}$$

$$C_{31} = \begin{bmatrix} A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ B_{41} \end{bmatrix}$$

$$C_{31} = A_{31} \cdot B_{11} + A_{32} \cdot B_{21} + A_{33} \cdot B_{31} + A_{43} \cdot B_{41}$$

Blocked Contractions

Visualization of $\mathcal{A} \star \mathcal{B}$ where "$\star$" is Some Contraction



Block Unfoldings and Block Contractions

The Setting...

Compute

$$\mathcal{C} \;=\; \mathcal{A} \star \mathcal{B}$$

where $\star$ is some contraction and $\mathcal{A}$ and $\mathcal{B}$ are blocked conformably.

Results

We have shown how to frame this as a block matrix product...

$$\mathcal{C}_{R\times C} \;=\; \mathcal{A}_{R\times\Lambda} \cdot \mathcal{B}_{\Lambda\times C}$$

Ongoing...

High-Performance Implementations, Data Structures, Strassen Ideas, Communication Lower Bounds

Block Representation and Approximation

Useful Representations

The Higher Order SVD

Compute the SVD of each modal unfolding and "glue together the results to characterize/approximate the original tensor.

The Higher-Order Kronecker Product SVD

Compute the KSVD of an arbitrary block unfolding and use the results to characterize/approximate the original tensor.

The Higher Order Singular Value Decomposition (HOSVD)

Basic Idea (Order-3 Case)

The HOSVD of an n_1-by-n_2-by-n_3 tensor $\mathcal{A}$ involves computing the matrix SVDs of its modal unfoldings $\mathcal{A}_{(1)}$, $\mathcal{A}_{(2)}$, and $\mathcal{A}_{(3)}$:

$$\begin{aligned} U_1^T \mathcal{A}_{(1)} V_1 &= \Sigma_1 \\ U_2^T \mathcal{A}_{(2)} V_2 &= \Sigma_2 \\ U_3^T \mathcal{A}_{(3)} V_3 &= \Sigma_3 \end{aligned}$$

permitting us to write

$$\mathcal{A} = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \sum_{j_3=1}^{n_3} \mathcal{S}(j_1, j_2, j_3) \cdot \left(U_1(:, j_1) \circ U_2(:, j_2) \circ U_3(:, j_3) \right)$$

De Lathauer, De Moor, and Vandewalle (2000)

The Higher Order Singular Value Decomposition (HOSVD)

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$$\begin{aligned} U_1^T \mathcal{A}_{(1)} V_1 &= \Sigma_1 \\ U_2^T \mathcal{A}_{(2)} V_2 &= \Sigma_2 \\ U_3^T \mathcal{A}_{(3)} V_3 &= \Sigma_3 \end{aligned}$$

permitting us to **approximate**

$$\mathcal{A} \approx \sum_{j_1=1}^{\mathbf{r_1}} \sum_{j_2=1}^{\mathbf{r_2}} \sum_{j_3=1}^{\mathbf{r_3}} \mathcal{S}(j_1, j_2, j_3) \cdot (U_1(:, j_1) \circ U_2(:, j_2) \circ U_3(:, j_3))$$

De Lathauer, De Moor, and Vandewalle (2000)

The Kronecker Product SVD (KPSVD)

For Uniformly Blocked Matrices...

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1q} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pq} \end{bmatrix} = \sum_{m=1}^{r} \sigma_m U_m \otimes V_m .$$

Nearest Sum of s Kronecker Products in the Frobenius Norm...

$$A_s \; = \; \sum_{m=1}^{s} \sigma_m U_m \otimes V_m .$$

Pitsianis and VL (1992)

Approximating a Special Order-4 Tensor

A Structured Summation that is $O(N^4)$

Compute

$$\mu = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N} \mathcal{A}(i,j,k,\ell) v_i v_j v_k v_\ell$$

where $v \in \mathbb{R}^N$ and $\mathcal{A}$ has the following symmetries:

$$\mathcal{A}(i,j,k,\ell) = \begin{cases} \mathcal{A}(j,i,k,\ell) \\ \mathcal{A}(i,j,\ell,k) \\ \mathcal{A}(k,\ell,i,j) \end{cases}$$

Three Symmetries

The [1 2] × [3 4] Unfolding Inherits these Symmetries...

280	206	100	206	182	187	100	187	296
206	328	188	182	138	148	187	244	143
100	188	176	187	148	122	296	143	326
206	182	187	328	138	244	188	148	143
182	138	148	138	312	192	148	192	212
187	148	122	244	192	272	143	212	200
100	187	296	188	148	143	176	122	326
187	244	143	148	192	212	122	272	200
296	143	326	143	212	200	326	200	280

The Kronecker Product SVD (KPSVD)

The KSVD is Structured...

$$\mathcal{A}_{[1\ 2]\times[3\ 4]} \;=\; \sum_{m=1}^{r} \sigma_m U_m \otimes U_m \qquad U_m^T = U_m$$

i.e.,

$$\mathcal{A}(i,j,k,l) \;=\; \sum_{m=1}^{r} \sigma_m U_m(i,j) U_m(k,\ell)$$

The summation μ becomes an $O(sN^2)$ summation:

$$\begin{aligned}\mu \;&=\; \sum_{m=1}^{r}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N} \sigma_m U_m(i,j) U_m(k,\ell) v_i v_j v_k v_\ell \\ &=\; \sum_{m=1}^{r} \sigma_m (v^T U_m v)^2 \;\approx\; \sum_{m=1}^{s} \sigma_m (v^T U_m v)^2\end{aligned}$$

The Higher Order KSVD

The Framework

$\mathcal{A}$ is a block tensor $(\mathcal{A}_{\mathbf{i}})$.

Let $\mathbf{A}$ be a block unfolding.

Compute its KSVD: $\mathbf{A} \;=\; \sum \sigma_m U_m \otimes V_m$

Equivalent to writing the block tensor $\mathcal{A}$ as a sum of "tensor Kronecker Products":

$$\mathcal{A} \;=\; \sum \sigma_m \,\mathcal{U}_m \text{``}\otimes\text{''}\, \mathcal{V}_m$$

If $\mathcal{U}$ and $\mathcal{V}$ are tensors, then $\mathcal{U}$ "$\otimes$ "$\mathcal{V}$ is a block tensor whose **i***-th block is the tensor $\mathcal{U}(\mathbf{i}) \cdot \mathcal{V}$, i.e., $\mathcal{U}(i_1, i_2, i_3) \cdot \mathcal{V}$.*

Summary

Block unfoldings preserve structure and locality of data.

The higher-order Kronecker Product SVD offers a block-level approach to low-rank tensor approximation.

The symmetric embedding shows how new algorithms and analyses are prompted by thinking at the "block" level.

In my opinion, blocking will eventually have the same impact in tensor computations as it does in matrix computations.