Models of KAT

In this lecture we show that the equational theories of KAT, KAT* (the star-continuous Kleene algebras with tests), and relational Kleene algebras with tests coincide. We also introduce a family of language-theoretic models consisting of regular sets of guarded strings, which play the same role in KAT that the regular sets play in Kleene algebra. These results are from [2].

The Language of Kleene Algebra with Tests

Let P and B be disjoint finite sets of symbols. Elements of P are called primitive actions and elements of B are called primitive tests. Terms and Boolean terms are defined inductively:

- any primitive action p is a term
- any primitive test b is a Boolean term
- 0 and 1 are Boolean terms
- if p and q are terms, then so are p + q, pq, and p* (suitably parenthesized if necessary)
- if b and c are Boolean terms, then so are b + c, bc, and b (suitably parenthesized if necessary)
- any Boolean term is a term.

The set of all terms over P and B is denoted RExp_{P,B}. The set of all Boolean terms over B is denoted RExp_{B}.

An interpretation over a Kleene algebra with tests $K$ is any homomorphism (function commuting with the distinguished operations and constants) defined on RExp_{P,B} and taking values in $K$ such that the Boolean terms are mapped to elements of the distinguished Boolean subalgebra.

If $K$ is a Kleene algebra with tests and $I$ is an interpretation over $K$, we write $K, I \models \varphi$ if the formula $\varphi$ holds in $K$ under the interpretation $I$ according to the usual semantics of first-order logic. We write KAT $\models \varphi$ (respectively, KAT* $\models \varphi$) if the formula $\varphi$ is a logical
consequence of the axioms of KAT (respectively, KAT∗). The only formulas we consider are equations or equational implications (universal Horn formulas).

We write KAT ⊨ ϕ if the formula ϕ is a logical consequence of KAT, i.e. if ϕ holds under all interpretations over Kleene algebras with tests. We write KAT∗ ⊨ ϕ if ϕ holds under all interpretations over star-continuous Kleene algebras with tests.

Guarded Strings

Let P and B be disjoint finite sets of symbols. Our language-theoretic model of Kleene algebras with tests is based on the idea of guarded strings over P and B. Guarded strings were introduced in [1].

We obtain a guarded string from a string x ∈ P∗ by inserting atoms interstitially among the symbols of x. An atom is a Boolean expression representing an atom (minimal nonzero element) of the free Boolean algebra on generators B.

Formally, an atom of $B = \{b_1, \ldots, b_k\}$ is a string of literals $c_1 c_2 \cdots c_k$, where each $c_i \in \{b_i, \overline{b_i}\}$. This assumes an arbitrary but fixed order $b_1 < b_2 < \cdots < b_k$ on B; for technical reasons, we require the literals in an atom to occur in this order. There are exactly $2^k$ atoms, and they are in one-to-one correspondence with the truth assignments to B. We denote atoms of B by $\alpha, \beta, \alpha_0, \ldots$. The set of all atoms of B is denoted $\text{Atoms}_B$. The set $\text{Atoms}_B$ will turn out to be the multiplicative identity of our language-theoretic model $\text{Reg}_{P,B}$.

If $b \in B$ and $\alpha$ is an atom of B, we write $\alpha \leq b$ if $b$ occurs positively in $\alpha$ and $\alpha \leq \overline{b}$ if $b$ occurs negatively in $\alpha$. This notation is consistent with the natural order in the free Boolean algebra generated by B.

Intuitively, the symbols of P can be thought of as instructions and atoms as conditions that must be satisfied at some point in the computation. If $\alpha \leq c_i$, then $\alpha$ asserts that $c_i$ holds (and $\overline{c_i}$ fails) at that point in the computation.

**Definition 13.1** A guarded string over P and B is any element of $(\text{Atoms}_B P)^* \text{Atoms}_B$; that is, any string

$$\alpha_0 p_1 \alpha_1 p_2 \cdots p_n \alpha_n, \quad n \geq 0,$$

where each $\alpha_i$ is an atom of B and each $p_i \in P$. Note that a guarded string begins and ends with an atom. If x is the guarded string above, we define first $x \overset{\text{def}}{=} \alpha_0$ and last $x \overset{\text{def}}{=} \alpha_n$. In the case $n = 0$, x is just a single atom, and first $x = \text{last } x$.

The set of all guarded strings over P and B is denoted $\text{GS}_{P,B}$, or just GS when P and B are understood.

Let $\overline{B} = \{\overline{b} \mid b \in B\}$. We denote strings in $(P \cup B \cup \overline{B})^*$, including guarded strings, by the letters x, y, z, $x_1, \ldots$. 

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The analog of concatenation for guarded strings is fusion product.

**Definition 13.2** The fusion product operation \( \cdot \) is a partial binary operation on GS defined as follows. If last \( x = \) first \( y \), then the fusion product \( xy \) exists and is equal to the string obtained by concatenating \( x \) and \( y \), but writing the common atom last \( x = \) first \( y \) only once between them.

For example, if \( B = \{b, c\} \) and \( P = \{p, q\} \), then
\[
bc\bar{p}bc \cdot \bar{b}cq\bar{c} = bc\bar{p}bcq\bar{c}.
\]
If last \( x \neq \) first \( y \), then the fusion product \( xy \) is undefined. We usually omit the \( \cdot \) in expressions. If \( A, B \subseteq GS \), define
\[
AB \overset{\text{def}}{=} \{xy \mid x \in A, y \in B, xy \text{ exists}\}.
\]
Thus \( AB \) consists of all existing fusion products of guarded strings in \( A \) with guarded strings in \( B \). For example, if \( B = \{b, c\} \), \( P = \{p, q\} \), and
\[
A = \{bc\bar{p}c, \bar{b}c, bcq\bar{c}\} \\
B = \{\bar{b}c\bar{p}bc, \bar{b}c, b\bar{c}qbc\}.
\]
then
\[
AB = \{bc\bar{p}\bar{c}pbc, bcp\bar{b}c, \bar{b}c, b\bar{c}qbc\}.
\]

Whereas the operation \( \cdot \) is partial when applied to guarded strings, it is total when applied to sets of guarded strings. Note that if there are no existing fusion products of strings from \( A \) and \( B \), then \( AB = \emptyset \). It is not difficult to show that \( \cdot \) is associative, that it distributes over union, and that it has two-sided identity \( \text{Atoms}_B \).

We now define a language-theoretic model \( \text{Reg}_{P,B} \) based on guarded strings. The elements of \( \text{Reg}_{P,B} \) will be the regular sets of guarded strings over \( P \) and \( B \) (although we have not yet defined regular in this context). We will also give a standard interpretation of terms in \( R\text{Exp}_{P,B} \) over \( \text{Reg}_{P,B} \) analogous to the standard interpretation of regular expressions as regular sets.

For \( A \subseteq GS \), define inductively
\[
A^0 \overset{\text{def}}{=} \text{Atoms}_B \\
A^{n+1} \overset{\text{def}}{=} A \cdot A^n.
\]
The asterate operation for sets of guarded strings is defined by
\[
A^* \overset{\text{def}}{=} \bigcup_{n \geq 0} A^n.
\]
Let $\overline{\cdot}$ denote set complementation in $\text{Atoms}_B$. That is, if $A \subseteq \text{Atoms}_B$, then $\overline{A} = \text{Atoms}_B - A$. Consider the structure

$$(2^{\text{GS}}, 2^{\text{Atoms}_B}, \cup, \cdot, \ast, \overline{\cdot}, \emptyset, \text{Atoms}_B),$$

which we denote briefly by $2^{\text{GS}}$. It is quite straightforward to verify that this is a star-continuous Kleene algebra with tests; that is, it is a model of $\text{KAT}^\ast$. The Boolean algebra axioms hold for $2^{\text{Atoms}_B}$ because it is a set-theoretic Boolean algebra.

The star-continuity condition follows immediately from the definition of $\ast$ and the distributivity of fusion product over infinite union. Since

$$B^* = \bigcup_{n \geq 0} B^n,$$

we have that

$$AB^*C = A \cdot (\bigcup_{n \geq 0} B^n) \cdot C = \bigcup_{n \geq 0} AB^n C.$$

Both of these expressions denote the set

$$\{xyz \mid x \in A, z \in C, \exists n y \in B^n\}.$$

For $p \in \mathbb{P}$ and $b \in B$, define

$$G(p) \overset{\text{def}}{=} \{\alpha\beta \mid \alpha, \beta \in \text{Atoms}_B\} \quad (13.1)$$

$$G(b) \overset{\text{def}}{=} \{\alpha \in \text{Atoms}_B \mid \alpha \leq b\}. \quad (13.2)$$

The structure $\text{Reg}_{\mathbb{P}, B}$ is defined to be the subalgebra of $2^{\text{GS}}$ generated by the elements $G(p)$ for $p \in \mathbb{P}$ and $G(b)$ for $b \in B$. Elements of $\text{Reg}_{\mathbb{P}, B}$ are called regular sets.

**Standard Interpretation**

The map $G$ defined on primitive actions and primitive tests in (13.1) and (13.2) extends uniquely by induction to a homomorphism $G : \text{RExp}_{\mathbb{P}, B} \to \text{Reg}_{\mathbb{P}, B}$:

$$G(p + q) \overset{\text{def}}{=} G(p) \cup G(q) \quad G(pq) \overset{\text{def}}{=} G(p) \cdot G(q)$$

$$G(1) \overset{\text{def}}{=} \text{Atoms}_B \quad G(\overline{b}) \overset{\text{def}}{=} \text{Atoms}_B - G(b)$$

$$G(0) \overset{\text{def}}{=} \emptyset \quad G(p^\ast) \overset{\text{def}}{=} G(p)^\ast.$$

The map $G$ is called the standard interpretation over $\text{Reg}_{\mathbb{P}, B}$. 

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Relational Models

Relational Kleene algebras with tests are interesting because they closely model our intuition about programs. In a relational model, the elements of $K$ are binary relations and $\cdot$ is interpreted as relational composition. Elements of the Boolean subalgebra are subsets of the identity relation.

Formally, a relational Kleene algebra with tests on a set $X$ is any structure

$$(K, B, \cup, \circ, *, \neg, \emptyset, \iota)$$

such that

$$(K, \cup, \circ, *, \emptyset, \iota)$$

is a relational Kleene algebra, i.e. $K$ is a family of binary relations on $X$, $\circ$ is ordinary relational composition, $*$ is reflexive transitive closure, and $\iota$ is the identity relation on $X$; and

$$(B, \cup, \circ, \neg, \emptyset, \iota)$$

is a Boolean algebra of subsets of $\iota$ (not necessarily the whole powerset).

All relational Kleene algebras with tests are star-continuous. We write $\text{REL} \models \varphi$ if the formula $\varphi$ holds in all relational Kleene algebras in the usual sense of first-order logic.

Completeness of $\text{KAT}^*$ over $\text{Reg}_{P, B}$

Now we show that an equation $p = q$ is a theorem of star-continuous Kleene algebra with tests iff it holds under the standard interpretation $G$ over $\text{Reg}_{P, B}$, where $P$ and $B$ contain all primitive action and test symbols, respectively, appearing in $p$ and $q$. Thus $\text{Reg}_{P, B}$ is the free Kleene algebra with tests on generators $P$ and $B$. In the next lecture, we will strengthen these results by removing the assumption of star-continuity.

**Theorem 13.3** Let $p, q \in \text{RExp}_{P, B}$. Then

$$\text{KAT}^* \models p = q \iff G(p) = G(q).$$

Equivalently, $\text{Reg}_{P, B}$ is the free star-continuous Kleene algebra with tests on generators $P$ and $B$.

The forward implication is easy, since $\text{Reg}_{P, B}$ is a star-continuous Kleene algebra. The converse is a consequence of the following lemma.
Lemma 13.4 For any star-continuous Kleene algebra with tests $K$, interpretation $I : \text{RExp}_{P,B} \to K$, and $p, q, r \in \text{RExp}_{P,B}$,

$$I(pqr) = \sup_{x \in G(q)} I(px r)$$

where the supremum is with respect to the natural order in $K$. In particular,

$$I(q) = \sup_{x \in G(q)} I(x).$$

This result is analogous to the same result for Kleene algebras proved in Lecture ?? and the proof is similar. Note that the star-continuity axiom is a special case.

We are most interested in the second statement, but there is a slight subtlety that requires the stronger first statement as the induction hypothesis. In addition to the existence of the supremum, the more general statement provides a kind of infinite distributivity law over existing suprema. The need for this arises mainly in the induction case for $\cdot$.

Proof of Lemma 13.4. We proceed by induction on the structure of $q$. The basis consists of cases for primitive tests, primitive actions, 0 and 1. We argue the case for primitive actions and primitive tests explicitly.

For a primitive action $q \in P$, recall that

$$G(q) = \{\alpha q \beta \mid \alpha, \beta \in \text{Atoms}_B\}.$$ 

Then

$$I(pqr) = I(p)I(1)I(q)I(1)I(r) = \sup \{I(p)I(\alpha)I(q)I(\beta)I(r) \mid \alpha, \beta \in \text{Atoms}_B\} = \sup \{I(p\alpha q \beta r) \mid \alpha, \beta \in \text{Atoms}_B\} = \sup \{I(px r) \mid x \in G(q)\}.$$ 

Finite distributivity was used in the second step.

For a primitive test $b \in B$, recall that

$$G(b) = \{\alpha \mid \alpha \leq b\}.$$ 

Then

$$I(pbr) = I(p)I(b)I(r) = \sup \{I(p)I(\alpha)I(r) \mid \alpha \leq b\} = \sup \{I(p\alpha r) \mid \alpha \leq b\} = \sup \{I(px r) \mid x \in G(b)\}.$$
Again, finite distributivity was used in the second step.

The induction step consists of cases for $+$, $\cdot$, $\ast$, and $\neg$. The cases other than $\cdot$ and $\neg$ are the same as in the proof of Theorem ?? of Lecture ??.

For the case $\cdot$, recall that
\[
G(qq') = G(q) \cdot G(q') = \{ y\alpha z \mid y\alpha \in G(q), \ \alpha z \in G(q') \}.
\]
Applying the induction hypothesis twice,
\[
I(pqq'r) = \sup \{ I(pqvr) \mid v \in G(q') \}
= \sup \{ \sup \{ I(puvr) \mid u \in G(q) \} \mid v \in G(q') \}
= \sup \{ I(puvr) \mid u \in G(q), v \in G(q') \}.
\]
The last step follows from a purely lattice-theoretic argument: if all the suprema in question on the left hand side exist, then the supremum on the right hand side exists and the two sides are equal.

Now
\[
\sup \{ I(puvr) \mid u \in G(q), v \in G(q') \}
= \sup \{ I(py\alpha \beta zr) \mid y\alpha \in G(q), \ \beta z \in G(q') \}
= \sup \{ I(py\alpha zr) \mid y\alpha \in G(q), \ \alpha z \in G(q') \}\tag{13.3}
= \sup \{ I(py\alpha zr) \mid y\alpha \in G(q), \ \alpha z \in G(q') \}
= \sup \{ I(px'r) \mid x \in G(qq') \}.
\]
The justification for step (13.3) is that if $\alpha \neq \beta$, then the product in $K$ is 0 and does not contribute to the supremum.

For the case $\neg$, recall that
\[
G(b) = \text{Atoms}_B - G(b) = \{ \alpha \mid \alpha \not\leq b \} = \{ \alpha \mid \alpha \leq \overline{b} \}.
\]
Then
\[
I(p\overline{b}r) = \sup \{ I(p\alpha r) \mid \alpha \leq \overline{b} \} = \sup \{ I(p\alpha r) \mid \alpha \in G(\overline{b}) \}.
\]

Proof of Theorem 13.3. If $\text{KAT}^* \models p = q$ then $G(p) = G(q)$, since $\text{Reg}_{P,B}$ is a star-continuous Kleene algebra with tests. Conversely, if $G(p) = G(q)$, then by Lemma 13.4, for any star-continuous Kleene algebra with tests $K$ and any interpretation $I$ over $K$, $I(p) = I(q)$. Therefore $\text{KAT}^* \models p = q$. ∎
Completeness over Relational Models

Finally we show completeness of $\text{KAT}^*$ over relational interpretations. It will suffice to construct a relational model isomorphic to $\text{Reg}_{P,B}$. This construction is similar to a construction we have seen before for Kleene algebra in Lecture ?? for regular sets.

For $A$ any set of guarded strings, define

$$h(A) \overset{\text{def}}{=} \{(x, xy) \mid x \in \text{GS}, \ y \in A\}.$$

**Lemma 13.5** The language-theoretic model $2^\text{GS}$ and its submodel $\text{Reg}_{P,B}$ are isomorphic to relational models.

**Proof.** We show that the function $h : 2^\text{GS} \to 2^\text{GS} \times \text{GS}$ defined above embeds $2^\text{GS}$ isomorphically onto a subalgebra of the Kleene algebra of all binary relations on GS.

It is straightforward to verify that $h$ is a homomorphism:

\[
\begin{align*}
  h(A \cup B) &= h(A) \cup h(B) \\
  h(AB) &= \{(z, zr) \mid z \in \text{GS, } r \in AB\} \\
  &= \{(z, zpq) \mid z \in \text{GS, } p \in A, \ q \in B\} \\
  &= \{(z, zp) \mid z \in \text{GS, } p \in A\} \\
  &\quad \circ \{(zp, zpq) \mid z \in \text{GS, } p \in A, \ q \in B\} \\
  &= \{(z, zp) \mid z \in \text{GS, } p \in A\} \circ \{(y, yq) \mid y \in \text{GS, } q \in B\} \\
  &= h(A) \circ h(B).
\end{align*}
\]

\[
\begin{align*}
  h(A^*) &= h(\bigcup_{n \geq 0} A^n) \\
  &= \bigcup_{n \geq 0} h(A)^n \\
  &= h(A)^* \\
  h(\text{Atoms}_B) &= \{(x, x\alpha) \mid x \in \text{GS, } \alpha \in \text{Atoms}_B\} \\
  &= \{(x, x) \mid x \in \text{GS}\} \\
  &= \iota \\
  h(0) &= \emptyset \\
  h(B) &= h(\{\alpha \mid \alpha \notin B\}) \\
  &= \{(x, x\alpha) \mid \alpha \notin B\} \\
  &= \{(y\alpha, y\alpha) \mid \alpha \notin B\} \\
  &= \iota - \{(y\alpha, y\alpha) \mid \alpha \in B\} \\
  &= \iota - h(B).
\end{align*}
\]
The function $h$ is injective, since $A$ can be uniquely recovered from $h(A)$:

$$A = \{ y | \exists \alpha \ (\alpha, y) \in h(A) \}.$$

The submodel $\text{Reg}_{P,B}$ is perforce isomorphic to a relational model on GS, namely the image of $\text{Reg}_{P,B}$ under $h$. \hfill \Box

Combining Theorem 13.3, Lemma 13.5, and the fact that all relational models are star-continuous Kleene algebras with tests, we have

**Theorem 13.6** Let $\text{REL}$ denote the class of all relational Kleene algebras with tests. Let $p, q \in \text{RExp}_{P,B}$. The following are equivalent:

(i) $\text{KAT}^* \models p = q$

(ii) $G(p) = G(q)$

(iii) $\text{REL} \models p = q$.

In the next lecture we will remove the assumption of star-continuity and show that the statement $\text{KAT} \models p = q$ can be added to this list. Thus $\text{KAT}$ is complete for the equational theory of relational models and $\text{Reg}_{P,B}$ forms the free $\text{KAT}$ on generators $P$ and $B$. This result is analogous to the completeness result of Lecture ??, which states that the regular sets over a finite alphabet $P$ form the free Kleene algebra on generators $P$.

**References**
