Korteweg-deVries Equation and Generalizations. VI. Methods for Exact Solution

by Gardner, Greene, Kruskal, and Miura

Presentation by Alan Demlow

CS 722, Spring 2000

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped–not so the mass of water in the channel which it had put in motion; it accumulated around the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth, and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel."
J.S. Russell, 1834

J.S. Russell first observed “waves of translation” (solitons) in 1834:

- Russell built a wave tank in his garden to study them.
- Observed that waves did not break up, spread out, or lose strength over distance.
- Made canal boats go faster by getting them to ride solitons.
- Russell was convinced of fundamental importance of solitons; contemporaries regarded them as a curiosity.
- Russell known instead for study of Doppler shift, ship hull design.
- Died relatively poor in 1882.

Several others investigated long waves and solitary waves in the period 1845-1895:

- Was generally believed that waves either “broke” or dispersed.
- “Solitary” waves were known to investigators.
- Either ignored, or thought to be stable only to an approximation.
- Perfect balance between tendency to disperse and tendency to break not thought possible, even in abscence of friction.

Korteweg and de Vries derived the KdV equation in 1895:

$$u_t + uu_x + u_{xxx} = 0$$

- Governs evolution of long-wavelength, small-amplitude waves in shallow water.
- K-dV knew two explicit solutions: cnoidal (oscillatory) and $u(x,t) = 3c^2 sech^2[\frac{1}{2}c(x - c^2t - x_0)]$.
- $sech$ solutions are "solitary": single "humps" die exponentially to 0 away from maximum.
- Solitary waves have speed proportional to height.
- K-dV thus showed that dispersive, breaking tendencies could indeed be balanced.
- Extensive 1945/46 obit of Korteweg in Annals of the Royal Dutch Academy didn't mention KdV equation!

Little further research until 1965, when Zabusky and Kruskal numerically investigated the KdV equation:

- KdV equation had been rederived in other areas of physics, thus further interest.
- For experiments, Z-K assumed periodic boundary conditions on $[0, 2]$.
- Initial conditions were $u(x, 0) = \cos \pi x$.
- Solution evolved into a train of 8 distinct waves.
- Waves interacted nonlinearly (i.e., non-additively) *but regained their original shape after such interactions!*
- Z-K were the first to use the term "soliton"; waves were so called because of their similarity to particles.

The observations made so far about solitons can be summed up in a definition:

A soliton is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interactions with other solitary waves, or more generally, with another arbitrary localized disturbance.

After Zabusky and Kruskal's experimental discovery, there was great interest in understanding solitons mathematically:

- Prior to this time, no general methods for studying nonlinear PDE's.
- Many researchers contributed to understanding the KdV equation during the period 1965-1974: Lax, Gardner, Miura, Greene, Kruskal, others.
- Resulted in the *Inverse Scattering Transform* (IST).
- IST is general method for solving nonlinear equations, akin to Fourier transform in linear context.

Solving $u_t(x,t) = u_{xx}(x,t)$, $u(x,0) = f(x)$ using Fourier transform requires three steps:

1. (Transform I.C.) Fourier-transform $u(x,0)$ in x:
$$\widehat{u}(\xi,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx.$$
2. (Find spatial transform of u(x,t)) FT turns differentiation into multiplication:
$$\widehat{\frac{\partial^k u}{\partial x^k}}(\xi,t) = (-i\xi)^k \widehat{u}(\xi,t).$$
Thus solve ODE $\widehat{u}_t(\xi,t) = -\xi^2 \widehat{u}(\xi,t)$:
$$\widehat{u}(\xi,t) = e^{-\xi^2 t}\widehat{u}(\xi,0) = e^{-\xi^2 t}\widehat{f}(\xi).$$
3. (Inverse transform) Invert the FT to find u:
$$u(x,t) = \int_{-\infty}^{\infty} e^{ix\xi} e^{-\xi^2 t}\widehat{f}(\xi)d\xi.$$

IST method also involves 3 basic steps. Step 1 is finding scattering data at $t = 0$ (*scattering transform*).

- Consider Schrödinger's Equation with potential u, eigenvalue λ:

$$\psi_{xx} - (u - \lambda)\psi = 0$$

  (motivated by Miura transformation).
- Find discrete spectrum of SE, i.e., find eigenvalues $\lambda_1 = -\kappa_1^2, ..., \lambda_N = -\kappa_N^2$.
- Known from quantum mechanics that normalized eigenfunctions ψ_n satisfy $\psi_n(x,t) \cong c_n(t)e^{-\kappa_n x}$ as $x \to \infty$.

Step 1 of IST (continued):

- Next study continuous spectrum $\lambda = k^2$ of SE with eigenstates ψ_k (the ψ_k have unbounded L^2 norm).
- There exist *reflectance* $R(k)$ and *transmittance* $T(k)$ with

$$\begin{cases} \psi(x,t) \cong e^{-ikx} + R(k)e^{ikx}, & x \to +\infty, \\ \psi(x,t) \cong T(k)e^{-ikx}, & x \to -\infty. \end{cases}$$

- $|T|^2 + |R|^2 = 1$; in QM, $R(k)$ and $T(k)$ are how much of wave e^{-ikx} is reflected and transmitted as it comes from $+\infty$.
- Thus we have complete *scattering data* at $t = 0$: $R(k,0)$, $T(k,0)$, $\kappa_n(0)$, $c_n(0)$.

Step 2 of the IST method is to determine how the scattering data evolve with time. If the potential u evolves according to $u_t - 6uu_x + u_{xxx}$, then:

- The discrete eigenvalues $\lambda_n(t) = \lambda_n(0)$.
- $R(k,t) = R(k,0)e^{8ik^3t}$.
- $T(k,t) = T(k,0)$.
- $c_n(t) = c_n(0)e^{4\kappa_n^3 t}$.

Step 3 of the IST method is to find the potential u at time t given the scattering data at time t (inverse scattering transform).

- The inverse transform is hardest part of IST method.
- First solve integral equation

$$K(x,y) + B(x+y) + \int_z^\infty B(y+z)K(x,z)dz = 0$$

where

$$B(\xi) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n \xi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)e^{ik\xi}dk.$$

- Then $u(x,t) = -2\frac{d}{dx}K(x,x)$.
- All steps of the IST method have a unique solution, leading to unique solution of KdV.

Further analysis using IST method yields explanation of interaction of solitons:

- Assume a reflectionless potential, i.e., $R(k) = 0$ and

$$B(\xi) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n \xi}.$$

- Separation of variables, further manipulations yield

$$u = -2\frac{d^2}{dx^2} \log \det(I + C)$$

  where C depends on c_i's, κ_i's, x, and $C + I$ is positive definite.

Interaction of Solitons (continued)

- Further analysis yields

$$u(x,t) = -4 \sum_{n=1}^{N} \kappa_n \psi_n^2(x,t).$$

- We call each term $-4\kappa_n\psi_n^2$ a soliton.
- Thus each discrete eigenvalue of the Schrödinger equation with potential u corresponds to exactly one soliton!
- Also can show that interaction of solitons asymptotically results only in a shift of position: larger solitons shift to the right, smaller solitons shift to the left.

In summary, work of Gardner, Greene, Kruskal and Miura resulted in:

- A method to explicitly solve the KdV equation.
- A full explanation of nonlinear interaction of solitons which evolve according to KdV.
- Most importantly, a method by which many other kinds of nonlinear equations can be investigated (no research dead end like K-dV's paper).

The IST method has been applied to a number of other nonlinear equations. Ablowitz and Clarkson list:

- Ordinary differential equations
- PDE's in one spatial dimension
- Differential-difference equations
- Singular integro-differential equations
- PDE's in two spatial dimensions
- Multidimensional scattering equations
- Multidimensional differential geometric equations

Solitons in Physical Applications

- Internal solitons have been observed between thermal layers in seas in the Far East.
- Waves carry great energy and have caused problems with deep-sea oil drilling.
- One rig apparently was spun 90 degrees and moved 100 feet by a soliton!
- Wave packets have been followed for more than 250 miles over a period of 2 days.
- KdV also proposed as a model for Rossby waves (long waves between atmospheric layers).
- Solitons have appeared in plasmas, protein models, magna flow, high-energy physics, solid-state physics.

Solitons in fiber optics:

- Information translated into light pulses, transmitted through optical fibers over long distances.
- Use of soliton pulses proposed in 1973.
- Propagation of solitons in optical fibers demonstrated at Bell Labs in 1980.
- In 1992, Bell Labs achieved transmission of 5 gigabits/second over 15,000km and 10 gigabits/second over 11,000 km

A bit about the authors.

- Clifford Gardner: Ph.D. NYU '53. Currently Prof. Emeritus of Mathematics at UT-Austin, inactive for some time.
- John Greene: Physicist; last institutional address was UT-Austin, 1997.
- Martin D. Kruskal: Prof. Emeritus of Mathematics at Princeton, Prof. of Mathematics at Rutgers. Active in applied math/physics research, origami.
- Robert M. Miura: Ph.D. Princeton. Professor of Mathematics at University of British Columbia. Interests in mathematical biology.

A Very Short Bibliography:

Ablowitz and Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*

Discussion Questions

- Why was there such a long break in research on the KdV equation?
- Why are general methods for solving partial differential equations so hard to come by? Will we continue to see more general methods such as the IST and FT developed?
- Any experiences with solitons in research or otherwise?