We have argued in previous lectures that the equational theory of each of the following classes of interpretations is contained in the next in the list: Kleene algebras, star-continuous Kleene algebras, closed semirings, S-algebras, N-algebras, relational models, language models, $\mathsf{Reg}\,\Sigma$, and $\mathsf{Reg}\,\Sigma$ under the single canonical interpretation R_Σ.

In this lecture we show that all these classes from star-continuous Kleene algebra onward have the same equational theory. It suffices to show that any equation that holds under the canonical interpretation $R_\Sigma : \mathsf{Exp}\,\Sigma \to \mathsf{Reg}\,\Sigma$ holds in all star-continuous Kleene algebras. Later on we will add Kleene algebras to this list as well.

Lemma 1. *For all regular expressions $s, t, u \in \mathsf{Exp}\,\Sigma$, the following property holds in any star-continuous Kleene algebra K:*

$$stu = \sup_{x \in R_\Sigma(t)} sxu.$$

In other words, if K is star-continuous, then under any interpretation $I : \mathsf{Exp}\,\Sigma \to K$, the supremum of the set

$$\{I(sxu) \mid x \in R_\Sigma(t)\}$$

exists and is equal to $I(stu)$.

In particular, in the special case $s = u = 1$,

$$t = \sup_{x \in R_\Sigma(t)} x.$$

This says that if we take the regular set of strings $x \in \Sigma^*$ denoted by the regular expression t and interpret each such string individually in K under the map I, then the resulting set of elements of K has a supremum, and that supremum is the interpretation of t in K under I.

The star-continuity axiom itself is a special case of the lemma. It states that

$$ab^*c = \sup_{n \geq 0} ab^n c = \sup_{x \in R_\Sigma(b^*)} axc,$$

that is, K is star-continuous if under any interpretation $I : \mathsf{Exp}\,\{a, b, c\} \to K$, the supremum of the set $\{I(ab^n c) \mid n \geq 0\}$ exists and is equal to $I(ab^*c)$. This is the special case for $s = a$, $t = b^*$, and $u = c$. The lemma expresses a natural extension of the star-continuity property to all expressions s, t, u. Later on we will do the same thing with the axioms of Kleene algebra; there we will extend the axioms, which give the ability to solve one linear affine inequality, to a theorem that gives the solution to any finite system of linear affine inequalities.

Proof. Let K be an arbitrary star-continuous Kleene algebra. We proceed by induction on the structure of t. There are three base cases, corresponding to the regular expressions $a \in \Sigma$, 1, and 0. For $a \in \Sigma$, we have $R_\Sigma(a) = \{a\}$ and

$$\sup_{x \,\in\, R_\Sigma(a)} sxu = sau,$$

since the supremum of a singleton set is just the unique element of that set. The case of 1 is similar, since $R_\Sigma(1) = \{\varepsilon\}$. Finally, since $R_\Sigma(0) = \varnothing$ and since 0 is the least element in K and therefore the supremum of the empty set,

$$\sup_{x \,\in\, R_\Sigma(0)} sxu \;=\; \sup \varnothing \;=\; 0 \;=\; s0u.$$

There are three cases to the inductive step, one for each of the operators $+$, $\cdot$, *. We give a step-by-step argument for the case $+$, followed by a justification of each step.

$$\begin{align}
s(t_1+t_2)u &= st_1u + st_2u \tag{1}\\
&= \sup_{x\,\in\,R_\Sigma(t_1)} sxu + \sup_{y\,\in\,R_\Sigma(t_2)} syu \tag{2}\\
&= \sup_{z\,\in\,R_\Sigma(t_1)\cup R_\Sigma(t_2)} szu \tag{3}\\
&= \sup_{z\,\in\,R_\Sigma(t_1+t_2)} szu. \tag{4}
\end{align}$$

Equation (1) follows from the distributive laws of Kleene algebra; (2) follows from the induction hypothesis on t_1 and t_2; (3) follows from the general property of upper semilattices that if A and B are two sets whose suprema $\sup A$ and $\sup B$ exist, then the supremum of $A\cup B$ exists and is equal to $\sup A+\sup B$ (this requires proof—see below); finally, equation (4) follows from the fact that the interpretation R_Σ is a homomorphism.

The general property used in step (3) states that if A and B are two subsets of an upper semilattice whose suprema $\sup A$ and $\sup B$ exist, then the supremum $\sup(A \cup B)$ exists and is equal to $\sup A + \sup B$. To prove this, we must show two things:

(i) $\sup A + \sup B$ is an upper bound for $A \cup B$; that is, for any $x \in A \cup B$, $x \leq \sup A + \sup B$; and

(ii) $\sup A + \sup B$ is the least such upper bound; that is, for any other upper bound y of the set $A \cup B$, $\sup A + \sup B \leq y$.

We can use the fact that in an upper semilattice with finitary join operation $+$, $x + y$ is the supremum of the set $\{x, y\}$. This holds in all Kleene algebras, since every KA is an upper semilattice with respect to $+$.

To show (i),

$$\begin{aligned}
x \in A \cup B &\Rightarrow x \in A \text{ or } x \in B\\
&\Rightarrow x \leq \sup A \text{ or } x \leq \sup B\\
&\Rightarrow x \leq \sup A + \sup B.
\end{aligned}$$

To show (ii), let y be any other upper bound for $A \cup B$. Then

$$\begin{aligned}
(\forall x \in A \cup B\ x \leq y) &\Rightarrow (\forall x \in A\ x \leq y) \text{ and } (\forall x \in B\ x \leq y)\\
&\Rightarrow \sup A \leq y \text{ and } \sup B \leq y\\
&\Rightarrow \sup A + \sup B \leq y.
\end{aligned}$$

Now we give a similar chain of equalities for the case of the multiplication operator $(\cdot)$.

$$\begin{align}
s(t_1t_2)u = st_1(t_2u) = \sup_{x\,\in\,R_\Sigma(t_1)} sx(t_2u) = \sup_{x\,\in\,R_\Sigma(t_1)} (sx)t_2u &= \sup_{x\,\in\,R_\Sigma(t_1)}\ \sup_{y\in R_\Sigma(t_2)} sxyu \notag\\
= \sup_{x\,\in\,R_\Sigma(t_1),\ y\,\in\,R_\Sigma(t_2)} sxyu \tag{5}\\
= \sup_{z\,\in\,R_\Sigma(t_1t_2)} szu. \notag
\end{align}$$

The inferences in the first line are justified by associativity of multiplication and the induction hypothesis. Step (5) uses a general property of suprema that is a stronger form of the property used in step (3) of the previous argument for $+$. It states that if $\mathcal{C}$ is a collection of subsets A of an ordered set, each with a supremum $\sup A$, and if $\sup_{A\in\mathcal{C}} \sup A$ exists, then $\sup \bigcup \mathcal{C}$ exists and is equal to $\sup_{A\in\mathcal{C}} \sup A$. The proof is a direct generalization of the proof for the weaker version above. The last step is by the definition of R_Σ.

Finally, for the case *, we have

$$st^*u = \sup_{n\geq 0} st^n u = \sup_{n\geq 0} \sup_{x\in R_\Sigma(t^n)} sxu = \sup_{x\,\in\,\bigcup_{n\geq 0} R_\Sigma(t^n)} sxu = \sup_{x\,\in\,R_\Sigma(t^*)} sxu.$$

This argument uses the axiom of star-continuity in the first step, the induction hypothesis in the second step, the same general property of suprema used above in the third step, and the definition of R_Σ in the last step. $\square$

Theorem 2. *Let s, t be regular expressions over Σ. The equation $s = t$ holds under all interpretations in all star-continuous Kleene algebras if and only if $R_\Sigma(s) = R_\Sigma(t)$.*

Proof. The forward implication is immediate, since $\mathsf{Reg}\,\Sigma$ is a star-continuous Kleene algebra. Conversely, by two applications of Lemma 1, if $R_\Sigma(s) = R_\Sigma(t)$, then under any interpretation in any star-continuous Kleene algebra,

$$s = \sup_{x\,\in\,R_\Sigma(s)} x = \sup_{x\,\in\,R_\Sigma(t)} x = t.$$ $\square$

Free Algebras

We have shown that the equational theory of the star-continuous Kleene algebras coincides with the equations true in $\mathsf{Reg}\,\Sigma$ under the interpretation R_Σ. Another way of saying this is that $\mathsf{Reg}\,\Sigma$ is the *free star-continuous Kleene algebra on generators* Σ. The term *free* intuitively means that $\mathsf{Reg}\,\Sigma$ satisfies only those equations that it is forced to satisfy in order to be a star-continuous Kleene algebra and no others.

Formally, a member A of a class of algebraic structures $\mathcal{C}$ of the same signature is said to be *free on generators X for the class $\mathcal{C}$* if

- A is generated by X;
- any function h from X into another algebra $B \in \mathcal{C}$ extends to a homomorphism $\widehat{h} : A \to B$.

The extension is necessarily unique, since a homomorphism is completely determined by its action on a generating set.

Thus to say that $\mathsf{Reg}\,\Sigma$ is the free star-continuous Kleene algebra on generators Σ says that $\mathsf{Reg}\,\Sigma$ is generated by Σ (actually, by $\{\{a\} \mid a \in \Sigma\} = \{R_\Sigma(a) \mid a \in \Sigma\}$), and for any star-continuous Kleene algebra K and map $h : \Sigma \to K$, there is a unique homomorphism $\widehat{h} : \mathsf{Reg}\,\Sigma \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Reg}\,\Sigma \arrow[dr, "\widehat{h}"] & \\
& K \\
\Sigma \arrow[uu, "R_\Sigma"] \arrow[ur, "h"'] &
\end{tikzcd} \tag{6}$$

In other words, $\widehat{h} \circ R_\Sigma = h$. This is an example of an *adjunction*, which we will describe in more depth later. The set function R_Σ that embeds the set Σ into the carrier of the algebra $\mathsf{Reg}\,\Sigma$ is called the *unit* of the adjunction.

Free algebras, if they exist, are unique up to isomorphism. If A and B are free algebras on generators X for a class $\mathcal{C}$, let $i : X \to A$ and $j : X \to B$ be the units of the two adjunctions. Since both algebras are free, these set maps extend to homomorphisms $\widehat{\imath} : B \to A$ and $\widehat{\jmath} : A \to B$, respectively.

$$\begin{tikzcd} A \arrow[dr, "\widehat{\jmath}"] & \\ & B \\ X \arrow[uu, "i"] \arrow[ur, "j"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[dr, "\widehat{\imath}"] & \\ & A \\ X \arrow[uu, "j"] \arrow[ur, "i"'] & \end{tikzcd}$$

Thus $\widehat{\imath} \circ j = i$ and $\widehat{\jmath} \circ i = j$. Then $(\widehat{\jmath} \circ \widehat{\imath}) \circ j = \widehat{\jmath} \circ (\widehat{\imath} \circ j) = \widehat{\jmath} \circ i = j$, so $\widehat{\jmath} \circ \widehat{\imath}$ and the identity function agree on the set $\{j(x) \mid x \in X\}$. But B is generated by this set, which says that $\widehat{\jmath} \circ \widehat{\imath} : B \to B$ is the identity morphism—it agrees with the identity on the generating set $\{j(x) \mid x \in X\}$, and homomorphisms are uniquely determined by their action on a generating set. Symmetrically, $\widehat{\imath} \circ \widehat{\jmath} : A \to A$ is also the identity, so $\widehat{\imath}$ and $\widehat{\jmath}$ are inverses, therefore A and B are isomorphic.

Congruence Relations and the Quotient Construction

Any class of algebras defined by universal equations or equational implications, even infinitary ones, has free algebras. Recall that the axioms for star-continuous Kleene algebra are of this form:

$$xy^n z \le xy^* z, \ n \ge 0 \qquad\qquad (\bigwedge_{n \ge 0} (xy^n z \le w)) \Rightarrow xy^* z \le w.$$

There is a general technique for constructing free algebras called a *quotient construction*. Here we give a brief account of this construction and how to apply it to obtain free algebras.

Fix an algebraic signature F consisting of some function symbols and their arities. Let A be any F-algebra. A binary relation $\equiv$ on A is call a *congruence* if

(i) $\equiv$ is an equivalence relation (reflexive, symmetric, transitive);

(ii) $\equiv$ is preserved by all the distinguished operations of the signature; that is, if f is n-ary, and if $x_i \equiv y_i$, $1 \le i \le n$, then $f(x_1, \ldots, x_n) \equiv f(y_1, \ldots, y_n)$. For example, for the signature of Kleene algebra, this would mean

$$\begin{aligned} x_1 \equiv y_1 \wedge x_2 \equiv y_2 &\Rightarrow x_1 + x_2 \equiv y_1 + y_2 \\ x_1 \equiv y_1 \wedge x_2 \equiv y_2 &\Rightarrow x_1 x_2 \equiv y_1 y_2 \\ x \equiv y &\Rightarrow x^* \equiv y^*. \end{aligned}$$

The *kernel* of a homomorphism $h : A \to B$ is

$$\ker h \stackrel{\text{def}}{=} \{(x, y) \mid h(x) = h(y)\}.$$

One can show that the kernel of any homomorphism with domain A is a congruence on A (Exercise **??**).

Conversely, given any congruence $\equiv$ on A, one can construct an F-algebra B and an epimorphism $h : A \to B$ such that $\equiv$ is the kernel of h. This is called the *quotient construction*. For any $x \in A$, define

$$[x] \stackrel{\text{def}}{=} \{y \in A \mid x \equiv y\},$$

the *congruence class* of x. Let

$$A/{\equiv} \stackrel{\text{def}}{=} \{[x] \mid x \in A\}.$$

One can make this into an F-algebra by defining

$$f^{A/\equiv}([x_1], \ldots, [x_n]) \stackrel{\text{def}}{=} [f^A(x_1, \ldots, x_n)].$$

The properties of congruence ensure that $f^{A/\equiv}$ is well defined and that the map $x \mapsto [x]$ is an epimorphism $A \to A/{\equiv}$.

Free Algebras as Quotients

Now we apply the quotient construction to obtain free algebras. As previously observed, if F is any signature, the set of well-formed terms TX over variables X can be regarded as an F-algebra in which the operations of F have their syntactic interpretation. For the signature of Kleene algebra, we have been calling the variables Σ, and the terms TX are the regular expressions $\mathsf{Exp}\,\Sigma$.

Let Δ be a set of equations or equational implications over TX, and let $\mathrm{Mod}\,\Delta$ denote the class of *models* of Δ, that is, the class of F-algebras that satisfy Δ. For example, if Δ consists of the axioms of star-continuous Kleene algebra, then $\mathrm{Mod}\,\Delta$ will be the class of all star-continuous Kleene algebras.

Let $\equiv$ be the smallest congruence on terms in TX containing all substitution instances of equations in Δ and closed under all substitution instances of equational implications in Δ. The relation $\equiv$ can be built inductively, starting with the substitution instances of equations in Δ and the reflexivity axiom $s \equiv s$ and adding pairs $s \equiv t$ as required by the substitution instances of equational implications in Δ and the symmetry, transitivity, and congruence rules.

One can now show that the quotient $TX/{\equiv}$ is the free $\mathrm{Mod}\,\Delta$ algebra on generators X. For any algebra A satisfying Δ and any map $h : X \to A$, h extends uniquely to a homomorphism $TX \to A$, which we also denote by h. The kernel of h is the set of equations satisfied by A under interpretation h. Since A satisfies Δ, the kernel of h contains all the equations of Δ and is closed under the equational implications of Δ. Since $\equiv$ is the smallest such congruence, $\equiv$ refines (is contained in) $\ker h$. Thus we can define $\widehat{h}([x]) \stackrel{\mathrm{def}}{=} h(x)$ and the resulting map $\widehat{h}$ will be well defined. This is the desired homomorphism $TX/{\equiv} \to A$.

For star-continuous Kleene algebra, the free algebra given by the quotient construction is $\mathsf{Exp}\,\Sigma/{\equiv}$, where $\equiv$ is the smallest congruence containing all substitution instances of the axioms of idempotent semirings, e.g. $s + (t + u) \equiv (s + t) + u$ for all $s, t, u \in \mathsf{Exp}\,\Sigma$, etc., and $st^n u \le st^* u$ for all $s, t, u \in \mathsf{Exp}\,\Sigma$ and $n \ge 0$ (where $s \le t$ is an abbreviation for $s + t \equiv t$), and contains $st^* u \le w$ whenever $st^n u \le w$ for all $n \ge 0$.

We have shown in Theorem 2 that $\mathsf{Reg}\,\Sigma$ is the free star-continuous KA on generators Σ, thus $\mathsf{Reg}\,\Sigma$ is isomorphic to $\mathsf{Exp}\,\Sigma/{\equiv}$. To see this directly, let $h : \Sigma \to K$ be an arbitrary function into a star-continuous Kleene algebra K, and extend h to a homomorphism $h : \mathsf{Exp}\,\Sigma \to K$. By Theorem 2, the set of equations that hold under the interpretation R_Σ, which is $\ker R_\Sigma$, is contained in the set of equations that hold under any interpretation in any star-continuous Kleene algebra, including h. Thus $\ker R_\Sigma$ refines $\ker h$. This says that we can define $\widehat{h}(R_\Sigma(s)) \stackrel{\mathrm{def}}{=} h(s)$, and the resulting map $\widehat{h} : \mathsf{Reg}\,\Sigma \to K$ will be well defined. This is the desired homomorphism making the diagram (6) commute.

Relations Among Algebras

The notion of free algebra is an example of a more general phenomenon called *adjunction*. An adjunction is a way of describing a particular relationship between categories of algebraic structures.

There are many examples of adjunctions in mathematics, but one very common occurrence is when some category C of algebras has more structure than another category D, and there is a canonical way to extend any D-algebra to a C-algebra. The construction normally constitutes a functor $F : \mathsf{D} \to \mathsf{C}$ called the *left adjoint* of the adjunction. There is normally a corresponding *forgetful functor* $G : \mathsf{C} \to \mathsf{D}$ going in the opposite direction that ignores the extra structure, called the *right adjoint*.

For example, every closed semiring is a star-continuous Kleene algebra, because the * operation can be defined in terms of the countable supremum operation and the star-continuity condition follows from the axioms of closed semirings. This constitutes a forgetful functor from the category of closed semirings, the category with more structure, to the category of star-continuous Kleene algebras, the category with less structure. Typically, forgetful functors do not modify the sets in any way, they just ignore some structure.

In the other direction, not every star-continuous KA is a closed semiring—the regular sets are not—but it turns out that every star-continuous KA can be extended to a closed semiring in a canonical way. For $\mathsf{Reg}\,\Sigma$, this construction would give 2^{Σ^*}, the closed semiring of all subsets of Σ^*.

We will show that adjunctions characterize the relationships among the following categories:

- KA^*, the category of star-continuous Kleene algebras and Kleene algebra morphisms;
- CS, the category of closed semirings and ω-continuous semiring morphisms (semiring morphisms that preserve suprema of countable sets); and
- SA, the category of S-algebras and continuous semiring morphisms (semiring morphisms that preserve arbitrary suprema).

The construction of the free algebra on a set of generators is an example of such an adjunction in which the category with less structure is Set, the category with no structure at all.

Adjunction

Formally, let $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{C} \to \mathsf{D}$ be functors between two categories C and D. We think of D as the category with less structure. We write $F \dashv G$ and say that F is a *left adjoint* of G and that G is a *right adjoint* of F if for any D-algebra X and C-algebra A, there is a natural one-to-one correspondence between morphisms $h : X \to GA$ in D and morphisms $\widehat{h} : FX \to A$ in C.

$$
\begin{tikzcd}
\mathsf{C} \arrow[d, bend left, "G"] \\
\mathsf{D} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\widehat{h}"] & A \arrow[d, "G"] \\
X \arrow[u, "F"] \arrow[r, "h"] & GA
\end{tikzcd}
$$

Note that this illustration does not denote a commutative diagram! The horizontal arrows represent morphisms in D (lower tier) and C (upper tier), whereas the vertical arrows represent the actions of the functors F and G on objects of their respective domains.

By *natural* we just mean that the one-to-one correspondence between $h : X \to GA$ and $\widehat{h} : FX \to A$ plays nicely with composition with morphisms in the following sense: if $f : X' \to X$ and $g : A \to A'$, and if $k = Gg \circ h \circ f : X' \to GA'$, then $\widehat{k} = g \circ \widehat{h} \circ Ff : FX' \to A'$.

$$
\begin{tikzcd}
FX' \arrow[r, "Ff"'] \arrow[rrr, bend left, "\widehat{k} = g \circ \widehat{h} \circ Ff"] & FX \arrow[r, "\widehat{h}"'] & A \arrow[r, "g"'] \arrow[d, "G"] & A' \arrow[d, "G"] \\
X' \arrow[u, "F"] \arrow[r, "f"] \arrow[rrr, bend right, "k = Gg \circ h \circ f"'] & X \arrow[u, "F"] \arrow[r, "h"] & GA \arrow[r, "Gg"] & GA'
\end{tikzcd}
$$

There are some special morphisms that exist. When $A = FX$, the morphism $\eta_X : X \to GFX$ down below corresponding to the identity $\mathsf{id} : FX \to FX$ up top is called the *unit* of the adjunction. Similarly, when $X = GA$, the morphism $\varepsilon_A : FGA \to A$ up top corresponding to the identity $\mathsf{id} : GA \to GA$ down below is called the *counit* of the adjunction. It turns out that the one-to-one correspondence between h and $\widehat{h}$ is uniquely determined by η_X and ε_A:

$$
\begin{tikzcd}
FX \arrow[r, "\mathsf{id}"'] \arrow[rr, bend left, "\widehat{h}"] & FX \arrow[r, "\widehat{h}"'] \arrow[d, "G"] & A \arrow[d, "G"] \\
X \arrow[u, "F"] \arrow[r, "\eta_X"] \arrow[rr, bend right, "h = G\widehat{h} \circ \eta_X"'] & GFX \arrow[r, "G\widehat{h}"] & GA
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "Fh"'] \arrow[rr, bend left, "\widehat{h} = \varepsilon_A \circ Fh"] & FGA \arrow[r, "\varepsilon_A"'] & A \arrow[d, "G"] \\
X \arrow[u, "F"] \arrow[r, "h"] \arrow[rr, bend right, "h"'] & GA \arrow[u, "F"] \arrow[r, "\mathsf{id}"] & GA
\end{tikzcd}
$$

The left-hand diagram shows that $h = G\widehat{h} \circ \eta_X$ and the right-hand diagram shows that $\widehat{h} = \varepsilon_A \circ Fh$.

In all instances we will consider, G is a *forgetful functor*, which means that A and GA have the same carrier, and Gg is set-theoretically the same function as g; it just has less structure to preserve.

In the free construction of the last lecture, the two categories are the category KA^* of star-continuous Kleene algebras and Kleene algebra homomorphisms and the category Set of sets and set functions. The free construction Reg constructs the regular sets on a given set of generators. It is the left adjoint of the forgetful functor $|\cdot|$ that associates to every star-continuous Kleene algebra its carrier.

$$
\begin{tikzcd}
\mathsf{KA}^* \arrow[d, bend left, "|\cdot|"] \\
\mathsf{Set} \arrow[u, bend left, "\mathsf{Reg}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Reg}\,\Sigma \arrow[r, "\widehat{h}"] & K \arrow[d, "|\cdot|"] \\
\Sigma \arrow[u, "\mathsf{Reg}"] \arrow[r, "h"] & |K|
\end{tikzcd}
$$

Here the unit of the adjunction is the set function $R_\Sigma : \Sigma \to |\mathsf{Reg}\,\Sigma|$ given by $R_\Sigma(a) = \{a\}$. The counit can be viewed as an evaluation function $\mathsf{eval}_K : \mathsf{Reg}\,|K| \to K$ that evaluates regular expressions over $|K|$ in K.

Completion by Star-Ideals

As observed, every closed semiring (ω-complete idempotent semiring) B gives a star-continuous Kleene algebra KB by defining $x^* = \sum_n x^n$. Also, if $h : B \to B'$ is an ω-continuous semiring morphism, then h must preserve *, therefore is a Kleene algebra morphism $Kh : KB \to KB'$. This constitutes a forgetful functor $K : \mathsf{CS} \to \mathsf{KA}^*$.

Similarly, every S-algebra (complete idempotent semiring) S is a closed semiring GS, and every morphism of complete semirings is ω-continuous. This constitutes a forgetful functor $G : \mathsf{SA} \to \mathsf{CS}$.

In the other direction, we have seen that not every star-continuous Kleene algebra is a closed semiring—for example, the regular sets over a finite alphabet are not. However, it is possible to construct in a canonical way a closed semiring CK extending any star-continuous Kleene algebra K. Similarly, although not every closed semiring is an S-algebra, every closed semiring can be extended to one.

Furthermore, any Kleene algebra homomorphism $h : K \to K'$ extends naturally to an ω-continuous semiring morphism $Cg : CK \to CK'$, and every ω-continuous semiring morphism $g : B \to B'$ extends to a continuous semiring morphism $Sg : SB \to SB'$. The functors $C : \mathsf{KA}^* \to \mathsf{CS}$ and $S : \mathsf{CS} \to \mathsf{SA}$ are left adjoints to K

and G, respectively.

$$\begin{tikzcd}
\mathsf{SA} \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] \\
\mathsf{CS} \arrow[u, bend left, "S"] \arrow[d, bend left, "K"] \arrow[d, phantom, "\dashv"] \\
\mathsf{KA}^* \arrow[u, bend left, "C"]
\end{tikzcd}$$

The basic construction used here is known as *completion by star-ideals* and was used by Conway to extend a star-continuous Kleene algebra to an S-algebra [1, Theorem 1, p. 102]. Thus Conway's construction is equivalent to the composition $S \circ C$ and is left adjoint to the composition $K \circ G$. The construction C, which shows that every star-continuous Kleene algebra is embedded in a closed semiring, can be described as a completion by *countably generated* star-ideals.

Definition 3. (Conway [1]) *Let K be a star-continuous Kleene algebra. A* star-ideal *is a subset I of K such that*

(i) *I is nonempty,*

(ii) *I is closed under $+$,*

(iii) *I is closed downward under $\leq$,*

(iv) *if $ab^n c \in I$ for all $n \geq 0$, then $ab^* c \in I$.*

A nonempty set A *generates* a star-ideal I if I is the smallest star-ideal containing A. We write $\langle A \rangle$ to denote the star-ideal generated by A. A star-ideal is *countably generated* if it has a countable generating set. If A is a singleton $\{x\}$, we abbreviate $\langle \{x\} \rangle$ by $\langle x \rangle$. Such an ideal is called *principal with generator x*.

Let K be a star-continuous Kleene algebra. We define the closed semiring CK as follows. The elements of CK will be the countably generated star-ideals of K. For any countable set $\{I_n \mid n \geq 0\}$ of countably generated star-ideals, define

$$\sum_n I_n = \langle \bigcup_n I_n \rangle.$$

This ideal is countably generated, since if A_n is countable and generates I_n for $n \geq 0$, then $\bigcup_n A_n$ is countable and generates $\sum_n I_n$. The operator $\sum$ is associative, commutative, and idempotent, since $\bigcup$ is.

For any pair of elements I, J, define

$$I \cdot J = \langle \{ab \mid a \in A,\ b \in B\} \rangle.$$

This ideal is countably generated if I and J are, since

$$\langle A \rangle \cdot \langle B \rangle = \langle \{ab \mid a \in \langle A \rangle,\ b \in \langle B \rangle\} \rangle = \langle \{ab \mid a \in A,\ b \in B\} \rangle,$$

and $\{ab \mid a \in A,\ b \in B\}$ is countable if A and B are (Exercise **??**).

The ideal $\langle 0 \rangle = \{0\}$ is included in every ideal and is thus an additive identity. It is also a multiplicative annihilator:

$$\langle 0 \rangle \cdot I = \langle \{ab \mid a \in \langle 0 \rangle,\ b \in I\} \rangle = \langle \{ab \mid a \in \{0\},\ b \in I\} \rangle = \langle 0 \rangle.$$

The ideal $\langle 1\rangle$ is a multiplicative identity:

$$\begin{aligned}\langle 1\rangle \cdot I &= \langle\{ab \mid a \in \langle 1\rangle,\ b \in I\}\rangle \\ &= \langle\{ab \mid a \in \{1\},\ b \in I\}\rangle \quad \text{by Exercise } \textbf{??} \\ &= \langle I\rangle = I.\end{aligned}$$

Finally, the distributive laws hold:

$$\begin{aligned}I \cdot \sum_n J_n &= \langle\{ab \mid a \in I,\ b \in \sum_n J_n\}\rangle = \langle\{ab \mid a \in I,\ b \in \langle\bigcup_n J_n\rangle\}\rangle \\ &= \langle\{ab \mid a \in I,\ b \in \bigcup_n J_n\}\rangle \ \text{ by Exercise } \textbf{??} \\ &= \langle\bigcup_n\{ab \mid a \in I,\ b \in J_n\}\rangle = \langle\bigcup_n\langle\{ab \mid a \in I,\ b \in J_n\}\rangle\rangle \\ &= \langle\bigcup_n\{ab \mid a \in I,\ b \in J_n\}\rangle = \sum_n I \cdot J_n,\end{aligned}$$

and symmetrically.

We note that condition (iv) in the definition of star-ideal ensures that the * operator gives the same value in the extension as in the original algebra.

Closed Semirings and S-algebras

The other half of the factorization of Conway's construction embeds an arbitrary closed semiring into an S-algebra. In comparison to the previous construction, this construction is much less interesting. We give the main construction and omit formal details.

Recall that closed semirings and S-algebras are both idempotent semirings with an infinite supremum operator $\sum$. The only difference is that closed semirings allow only countable suprema, whereas S-algebras allow arbitrary suprema. Morphisms of closed semirings are the ω-continuous semiring morphisms and those of S-algebras are the continuous semiring morphisms.

To embed a given closed semiring B in an S-algebra SB, we complete B by ideals. An *ideal* is a subset $A \subseteq B$ such that

(i) A is nonempty,

(ii) A is closed downward under $\leq$,

(iii) A is closed under countable suprema.

Take SB to be the set of ideals of B with the following operations:

$$\sum_\alpha I_\alpha = \langle\bigcup_\alpha I_\alpha\rangle \qquad I \cdot J = \langle\{ab \mid a \in I,\ b \in J\}\rangle \qquad 0 = \langle 0\rangle \qquad 1 = \langle 1\rangle.$$

The arguments from here on are quite analogous to those of the previous section. As before, condition (iii) ensures that the construction does not introduce any new countable suprema; that is, countable suprema are the same in SB and B.

References

[1] John Horton Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, London, 1971. Dover edition, 2012.