Equational Horn Logic

This is a complete deductive system for universal equational Horn logic. Judgments are of the form

$$\Delta\,;\Gamma \vdash \varphi,$$

where:

- φ is an unquantified Horn formula of the form

  $$s_1 = t_1 \to s_2 = t_2 \to \cdots \to s_n = t_n \to s = t,$$

  where the s's and t's are terms over an algebraic signature;

- Γ is a list of unquantified equations $s = t$, and

- Δ is a list of universally quantified Horn formulas $\forall\overline{x}\ \varphi$, where $\overline{x} = x_1, \ldots, x_k$ is a list of all variables appearing in φ.

The judgment $\Delta\,;\Gamma \vdash \varphi$ means that φ holds under the premises Δ and Γ. We should think of Δ as a library of universally quantified axioms and theorems and Γ as a set of equational assumptions.

Quantification rules

$$\Delta, \forall\overline{x}\ \varphi\,;\Gamma \vdash \varphi[\overline{t}/\overline{x}] \qquad\qquad \frac{\Delta\,; \vdash \varphi}{\Delta, \forall\overline{x}\ \varphi\,; \vdash}$$

The left-hand rule says that any axiom or theorem in the library can be instantiated by substituting any terms for the variables. The meta-expression $\varphi[\overline{t}/\overline{x}]$ denotes the result of substituting the terms $\overline{t} = t_1, \ldots, t_k$ for the variables $\overline{x} = x_1, \ldots, x_k$ in φ. The terms $\overline{t}$ may also contain variables.

The right-hand rule says that if an unquantified Horn formula is derivable from the axioms without any equational assumptions, then it holds for all possible values of the variables, so we can quantify the variables universally and enter it as a theorem in the library.

Equality rules

$$\Delta\,;\Gamma \vdash s = s \qquad \frac{\Delta\,;\Gamma \vdash s = t}{\Delta\,;\Gamma \vdash t = s} \qquad \frac{\Delta\,;\Gamma \vdash s = t \qquad \Delta\,;\Gamma \vdash t = u}{\Delta\,;\Gamma \vdash s = u}$$

$$\frac{\Delta\,;\Gamma \vdash s_1 = t_1 \qquad \ldots \qquad \Delta\,;\Gamma \vdash s_n = t_n}{\Delta\,;\Gamma \vdash f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)}\ (f \text{ is } n\text{-ary})$$

These are the usual laws of equality: reflexivity, symmetry, transitivity, and congruence. A derived rule is substitution of equals for equals: for any term u,

$$\frac{\Delta\,;\Gamma \vdash s_1 = t_1 \qquad \ldots \qquad \Delta\,;\Gamma \vdash s_n = t_n}{\Delta\,;\Gamma \vdash u[\overline{s}/\overline{x}] = u[\overline{t}/\overline{x}]}$$

Implication rules

$$\frac{\Delta\,;\Gamma \vdash s = t \to \varphi \qquad \Delta\,;\Gamma \vdash s = t}{\Delta\,;\Gamma \vdash \varphi} \qquad\qquad \frac{\Delta\,;\Gamma, s = t \vdash \varphi}{\Delta\,;\Gamma \vdash s = t \to \varphi}$$

The left-hand rule is called *modus ponens* and allows you to assert the conclusion of an equational implication if you have proved the premise. The right-hand rule allows you to discharge a hypothesis: if under the assumption $s = t$ you can prove φ, then you have proven the implication $s = t \to \varphi$.

There is another associated rule that lets you assert an assumption:

$$\Delta\,;\Gamma, s = t \vdash s = t$$

Structural rules In addition there are structural rules of weakening and reordering that allow you to permute the elements of Δ and Γ and introduce new elements of Γ for free.