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# ON WEAK COMPLETENESS OF INTUITIONISTIC PREDICATE LOGIC

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**1.** Suppose the $r_i$-placed relation symbols $P_i$, $1 \leqq i \leqq k$, are all the non-logical constants occurring in the closed formula $\mathfrak{A}$, also written as $\mathfrak{A}(P_1, \ldots, P_k)$, of Heyting's predicate calculus (HPC). Then HPC is called *complete for* $\mathfrak{A}$ provided ($\mathfrak{A}$ is valid) $\rightarrow \vdash_{\mathrm{HPC}} \mathfrak{A}$, i.e.

$$(1) \qquad (D)(P_1^*)\ldots(P_k^*)\mathfrak{A}_D(P_1^*, \ldots, P_k^*) \rightarrow (Ep)\mathrm{Prov}(p, \ulcorner\mathfrak{A}\urcorner).$$

Here $D$ ranges over arbitrary species, and $P_i^*$ over arbitrary (possibly incompletely defined) subspecies of $D^{r_i}$; $\mathfrak{A}_D$ is obtained by restricting the individual variables of $\mathfrak{A}$ to $D$; $\mathrm{Prov}(m, n)$ is the (natural) proof predicate for HPC, i.e. (for a given Gödel numbering of HPC) $m$ is the Gödel number of a proof in HPC of the formula with number $n$; and $\ulcorner\mathfrak{A}\urcorner$ denotes the Gödel number of the formula $\mathfrak{A}$.[1]

HPC is called *weakly complete for* $\mathfrak{A}$ if the double negation of (1) holds.

As in Remark 1.1 of [12], we also consider a strengthened version (1′) of (1) where the $P_i^*$ are required to be completely defined, in particular do not depend on parameters over free choice sequences.

It is understood that the logical operations are used intuitionistically throughout the paper. We use the 'logical' symbols, which are formal objects of HPC, also as notation for logical operations in interpreted assertions such as (1). The letters $m$, $n$, $p$ are reserved for variables over natural numbers, $\alpha$ for free choice sequences, while $x, y, z, u, v, x_1, \ldots, x_n$ are used as (formal) variables of HPC, $P, Q, I, S$ as 'predicate' symbols of HPC.

**2.** The principal known completeness results are: HPC is strongly complete (a) in sense (1) for quantifier-free and prenex formulae (for which derivability in HPC is decidable), by [13], and (b) even in sense (1′) for 'inductive definition formulae' (for which derivability in HPC is not recursively decidable; cf. Remark 3.1 of [12]), by Remark 3 below. By Theorem 4 of [12], HPC is weakly complete, also in sense (1′), for all *negative*

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[1] Throughout this paper we are primarily interested in syntactic relations between sequences of formulae, and not in the arithmetic relations between their Gödel numbers. We use the latter in order to apply the familiar notions of a primitive recursive or recursive property of natural numbers to classes of formulae. Also, the Gödel numbering is needed if one considers proofs of (1) in systems of intuitionistic set theory. — This remark and some other expository improvements of an earlier version of this paper are due to Kleene, in addition to some substantial contributions of his which are noted in the text.

formulae, i.e. those which contain no existence or disjunction symbols and no unnegated prime formulae.[2] (See the note added at the end.)

The main new result of the present paper is this: There is a formula of HPC for which HPC cannot be proved to be weakly complete in sense (1) by any methods of intuitionistic mathematics so far formalized.

More generally, if a formal system $S$ admits, besides the intended interpretation of the logical constants introduced by Brouwer and Heyting, certain variants of Kleene's recursive realizability interpretations, then there is a formula $\mathfrak{A}$ of HPC for which the double negation of (1) cannot be proved in $S$. Furthermore there is such an $\mathfrak{A}$, independent of $S$, which is the negation of a prenex formula.[3] — Also this $\mathfrak{A}$ is valid for all completely defined predicates if all constructive number-theoretic functions are recursive; so, on this assumption (Church's thesis), HPC is not complete in sense (1′) for this $\mathfrak{A}$.

**3.** An important result used in our proof is Theorem 1 below, which is essentially due to Gödel: for each primitive recursive relation $A(n, \alpha)$, (strong) completeness of HPC (for a suitable formula depending on $A$) implies

$$(\alpha)_B \neg\neg(En)A(n, \alpha) \rightarrow (\alpha)_B(En)A(n, \alpha), \tag{2}$$

where the subscript $B$ means that the $\alpha$ are chosen from the full binary spread $B$ with values 0 and 1.[4] Consequently, weak completeness implies

$$(\alpha)_B \neg\neg(En)A(n, \alpha) \rightarrow \neg\neg(\alpha)_B(En)A(n, \alpha). \tag{3}$$

Gödel's result is particularly useful because intuitionistic *set theory*, where (1) is formulated, is less highly developed than *second-order arithmetic*, where (2) and (3) are formulated.

Conversely, his result shows that *the schema* (2) *is equivalent to the completeness of HPC*. For, by the analysis in [13], particularly p. 382 (c), of Beth's semantic construction of intuitionistic logic [1], the schema (3) implies weak completeness of HPC, and, by Theorem 2 of [12], (2) then implies strong completeness.

Analysis of Gödel's proof shows that strong completeness of HPC for

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[2] Cf. the erratum mentioned in the list of references. The uncorrected form of Theorem 4 of [12] is false: the formula $\neg[(x)\neg\neg P(x) \;\&\; \neg(x)P(x)]$ is not a theorem of HPC, but for each $\mathfrak{P}(n)$ of arithmetic which does not contain existence or disjunction symbols, $\neg[(n)\neg\neg\mathfrak{P}(n) \;\&\; \neg(n)\mathfrak{P}(n)]$ is provable (in intuitionistic arithmetic).

[3] Hence, by Theorem 7 of [12], $\vdash_{HPC} \mathfrak{A} \leftrightarrow \vdash_{CPC} \mathfrak{A}$, where CPC denotes the classical predicate calculus.

[4] Alternatively, $A$ may be regarded as a relation between natural numbers $n$ and free choice *predicates* $[\leftrightarrow \alpha(n) = 0]$ in the sense of [5], where truth values T and $\perp$ are chosen freely (from the species of truth values). Note that $\alpha$ is a variable over free choice sequences in the sense of [5] or [9], and *not* over *absolutely* free ones [13].

the *negative* fragment implies

$$\neg\neg(En)A(n) \rightarrow (En)A(n) \tag{4}$$

for each primitive recursive predicate $A(n)$.[5]

**4.** It should be noted that (4) is not at all plausible on the intended interpretation of the logical constants, and so it is plausible that HPC *cannot be proved to be strongly complete*. On the other hand, (3) is not so implausible, and may be provable on the basis of as yet undiscovered axioms which hold for the intended interpretation (but not for the realizability interpretations). So *the problem whether* HPC *is weakly complete is still open*.

Quite generally, the significance of formal non-derivability in the intuitionistic higher-order systems which have *so far* been explicitly formalized is limited because these systems are weak (a fact which is of interest also independently of our main topic). By Theorem 7 below, Kleene's system [9] (and related systems of intuitionistic analysis) is an *inessential extension* of the fragment of (Heyting's) first-order intuitionistic arithmetic which consists of prenex and negative formulae.[6] Since the proof of Theorem 7 is finitist, it provides a *finitist relative consistency* proof for Kleene's system relative to Heyting's. — This contrasts sharply with set-theoretic (classical) mathematics where new first-order theorems are formally derivable from evident properties of higher-type objects such as the comprehension principle of the theory of types (though the known *examples* of such theorems are of logical rather than arithmetic interest).

**5.** The difference between set-theoretic and intuitionistic higher-order mathematics just mentioned is not to be regarded as permanent. Specifically, although Kleene's system [9] includes a strong statement of Brouwer's *fan theorem* by the Axiom Schema (4) of [9] p. 286, it does not include *the principle of induction with respect to unsecured sequences* which is used in Brouwer's informal proof [2]. It is possible that some formulation of this principle is not valid for the realizability interpretation, although, as Kleene will show in his forthcoming monograph [10], at least one natural formulation is.

Another intuitionistic notion which is not covered by our non-derivability results is that of an *absolutely* free choice sequence [13]. Of course, (3) cannot be proved in the system FC of [13], since FC does not contain variables for (ordinary) free choice sequences at all. But the recursive realizability

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[5] This result was given in the abstract [14].

[6] I do not know whether Kleene's system is an inessential extension for the whole of Heyting's arithmetic.

interpretation does not apply to FC (cf. [13] p. 387), and so it is possible that a formalization which contains both a theory of absolutely and ordinary free choice sequences could escape our non-derivability result. Since the intended interpretation of the logical constants applies to FC, it is plausible that (4) is not derivable in FC. However, the problem of [13] p. 386 Remark 8.3 is still open.

**6. Gödel's argument.** In 1957, Gödel indicated (in conversation) a proof of Theorem 2 below by combining (i) his own reduction [3] p. 194 of the truth of statements $(n)A(n)$, for primitive recursive $A(n)$, to the *satisfiability* of a suitable formula $\mathfrak{A}_G$ of (classical or intuitionistic) first-order predicate logic, and (ii) the constructive proof of Herbrand's theorem (or the first $\varepsilon$-theorem of [6]) which shows, for a suitable primitive recursive $\rho$, that $\mathrm{Prov}(p, \ulcorner\neg\mathfrak{A}_G\urcorner) \to (En)[n \leqq \rho(p) \;\&\; \neg A(n)]$. The same idea also establishes Theorem 1. The argument below is simplified by the use of several suggestions of Kleene; in particular, we replace (i) by a reduction of the truth of $(En)A(n)$ to the *validity* of a suitable formula $\mathfrak{A}_K$.[7]

THEOREM 1. *For each primitive recursive relation $A(n, \alpha)$ between natural numbers and free choice sequences, there is a prenex formula $\mathfrak{A}$ of* HPC (*without function or individual symbols and without identity*) *such that* (i) *completeness of* HPC, *in sense* (1), *for $\neg\mathfrak{A}$ implies*

$$(\alpha)_B\neg\neg(En)A(n, \alpha) \to (\alpha)_B(En)A(n, \alpha),$$

(ii) *completeness of* HPC, *in sense* (1′), *for $\neg\mathfrak{A}$ implies*

$$(\alpha^0)_B\neg\neg(En)A(n, \alpha^0) \to (\alpha)_B(En)A(n, \alpha),$$

*where the subscript $B$ means that the $\alpha$ range over all free choice sequences taking values 0 or 1 only, and $\alpha^0$ range over all effectively defined number-theoretic functions taking values 0 or 1 only.*

PROOF. The finite axiom systems $\mathfrak{Z}$ and $\mathfrak{A}_1$ below are sufficient to 'compute' $A(n, \alpha)$ for any given $n$ and $\alpha$. The existence of such systems on the basis of CPC is familiar, but is set out again here to ensure that a *prenex* axiomatization is possible on the basis of HPC.

AXIOMS FOR IDENTITY, ZERO, SUCCESSOR. $\mathfrak{U}(I, Z, S)$ is the conjunction of the closures of the following (quantifier-free) formulae:

$$I(x, x),\; I(x, y) \to [I(x, z) \to I(y, z)],\; I(x, y) \to [Z(x) \to Z(y)],$$
$$I(x, y) \to [S(x, z) \to S(y, z)],\; I(x, y) \to [S(z, x) \to S(z, y)],$$
$$[Z(x) \;\&\; Z(y)] \to I(x, y),\; [S(x, y) \;\&\; S(x, z)] \to I(y, z);$$

[7] $\mathfrak{A}_G$ is of the form $\neg[\mathfrak{B}_G \to (Ex)P(x)]$, and $\mathfrak{A}_K$ of the form $[\mathfrak{B}_K \to (Ex)P(x)]$, where both $\mathfrak{B}_G$ and $\mathfrak{B}_K$ are finite axiomatizations of a fragment of arithmetic (on the basis of predicate logic). $\mathfrak{B}_G$ contains axioms (for the successor relation) which ensure that all models of $\mathfrak{A}_G$ contains infinitely many distinct objects, while $\mathfrak{B}_K$ does not.

$\mathfrak{E}(Z, S)$ is $(Ex)Z(x)$ & $(x)(Ey)S(x, y)$;

$\mathfrak{E}_p(Z, S)$ is $(Ex)(Ex_1)\dots(Ex_p)[Z(x)$ & $S(x, x_1)$ & $\dots$ & $S(x_{p-1}, x_p)$;

$\mathfrak{Z}$, or $\mathfrak{Z}(I, Z, S)$, is $\mathfrak{E}(Z, S)$ & $\mathfrak{U}(I, Z, S)$;

$\mathfrak{Z}_p$ is $\mathfrak{E}_p(Z, S)$ & $\mathfrak{U}(I, Z, S)$.

The remaining axioms $\mathfrak{A}_1$ are obtained by converting a primitive recursive definition (p.r.d.) of the characteristic function $f_k(n, \alpha)$, with parameter $\alpha$, of $A(n, \alpha)$ into an inductive definition of $A(n, \alpha)$ with the principal relation symbol $P_k(x, y)$. — Recall that a p.r.d. of $f_k$ consists of a finite sequence of equations where, for $q \leqq k$, the $q$-th equation has one of the following forms: $f_0(n) = \alpha(n)$; for $q > 0$, (i) $f_q(n) = 0$, (ii) $f_q(n) = n'$, (iii) $f_q(n_1, \dots, n_r) = n_i$, for some $i$, $i \leqq r$, (iv) $f_q(n_1, \dots, n_r) = f_s[f_{s_1}(n_1, \dots, n_r), \dots, f_{s_t}(n_1, \dots, n_r)]$ where $s < q$, $s_i < q$ for $1 \leqq i \leqq t$, (v) $f_q(0, m) = f_s(m)$, $f_q(n', m) = f_t[n, m, f_q(n, m)]$, $s < q$, $t < q$.

Axioms for the computation of $A(n, \alpha)$. We denote by $\mathfrak{A}_1$, or $\mathfrak{A}_1(I, Z, S, Q, P_0, P_1, \dots, P_k)$, the conjunction of the closures of the following formulae (axioms for the graphs of $f_0, \dots, f_k$): $[Q(x)$ & $Z(y)] \rightarrow P_0(x, y)$, $[\neg Q(x)$ & $Z(y)$ & $S(y, z)] \rightarrow P_0(x, z)$, and, according to the cases (i)–(v) in the $q$-th equation of the p.r.d. of $f_k$, (i) $Z(y) \rightarrow P_q(x, y)$, (ii) $S(x, y) \rightarrow P_q(x, y)$, (iii) $P_q(x_1, \dots, x_r, x_i)$, (iv) $[P_{s_1}(x_1, \dots, x_r, y_1)$ & $\dots$ & $P_{s_t}(x_1, \dots, x_r, y_t)$ & $P_s(y_1, \dots, y_t, y)] \rightarrow P_q(x_1, \dots, x_r, y)$, (v) $[Z(x)$ & $P_s(y, z)] \rightarrow P_q(x, y, z)$ and $[P_q(x, y, z)$ & $P_s(x, y, z, u)$ & $S(x, v)] \rightarrow P_q(v, y, u)$, and, finally, the identity axioms $I(x, y) \rightarrow [Q(x) \rightarrow Q(y)]$, $I(x, y) \rightarrow [P_q(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_r) \rightarrow P_q(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_r)]$ for all $q \leqq k$ and all $i$, $1 \leqq i \leqq r$.

Remark 1. If $\alpha$ is absent, i.e. for primitive recursive predicates $A(n)$ of natural numbers, we use instead of $\mathfrak{A}_1$ the axioms $\mathfrak{A}_0(I, Z, S, P_1, \dots, P_k)$ obtained from a p.r.d. of the characteristic function for $A(n)$. The difference between $\mathfrak{A}_0$ and $\mathfrak{A}_1$ is that the axioms of $\mathfrak{A}_1$ involving $Q$ and $P_0$ are absent.

Representation of numerical and existential assertions. Let $\mathfrak{P}$, or $\mathfrak{P}(Z, P_k)$, be $(Ex)(Ey)[Z(y)$ & $P_k(x, y)]$.

For each free choice sequence $\alpha$ from $B$, let $\mathfrak{F}_p{}^\alpha$ denote the conjunction of the following formulae:

$(Ex)[Z(x)$ & $Q(x)]$ if $\alpha(0) = 0$, $(Ex)[Z(x)$ & $\neg Q(x)]$ if $\alpha(0) = 1$, and, for all $s \leqq p$,

$(Ex)(Ex_1)\dots(Ex_s)[Z(x)$ & $S(x, x_1)$ & $\dots$ & $S(x_{s-1}, x_s)$ & $Q(x_s)]$ if $\alpha(s) = 0$,

$(Ex)(Ex_1)\dots(Ex_s)[Z(x)$ & $S(x, x_1)$ & $\dots$ & $S(x_{s-1}, x_s)$ & $\neg Q(x_s)]$ if $\alpha(s) = 1$.

Note that $\mathfrak{F}_p{}^\alpha$ is well-defined since, for each $p$, only the first $p$ values of $\alpha$ are needed for the construction of the formulae $\mathfrak{F}_p{}^\alpha$.

Lemma 1. *Given a primitive recursive relation* $A(n, \alpha)$ *there are primitive recursive functions* $\sigma(n)$ *and* $\pi(n)$, *depending on the p.r.d. of the characteristic* *function of* $A(n, \alpha)$, *such that* (*for* $\alpha$ *from* $B$)

$$A(n, \alpha) \rightarrow [(\mathfrak{Z}_{\sigma(n)} \;\&\; \mathfrak{A}_1 \;\&\; \mathfrak{F}^{\alpha}{}_{\pi(n)}) \rightarrow \mathfrak{P} \textit{ is valid}],$$

*and hence*,

$$A(n, \alpha) \rightarrow [(\mathfrak{Z} \;\&\; \mathfrak{A}_1 \;\&\; \mathfrak{F}^{\alpha}{}_{\pi(n)}) \rightarrow \mathfrak{P} \textit{ is valid}].$$

For, if $A(n, \alpha)$, then, from the inductive definition of $A$, $[Z(x) \;\&\; S(x, x_1) \;\&\; \ldots \;\&\; S(x_{m-1}, x_m) \;\&\; \mathfrak{F}_p{}^{\alpha}] \rightarrow P_k(x_n, x)$ can be derived by substitution and detachment only, if $p$ is a bound for the arguments of $\alpha$ and $m$ for the arguments and values of the functions $f_1, \ldots, f_k$ which are used in the computation of $f_k(n, \alpha)$. The functions $\sigma$ and $\pi$, with $p \leqq \sigma(n)$, $m \leqq \pi(n)$, are obtained from the corresponding functions $\sigma_q$, $\pi_q$, $q \leqq k$, as follows, corresponding to the p.r.d. of $A$: $\pi_0(n) = n$, $\sigma_0(0) = 1$, $\sigma_0(n') = n'$; for (i) and (ii) $\pi_q(n) = 0$; (i) $\sigma_q(n) = n$, (ii) $\sigma_q(n) = n'$, (iii) $\sigma_q(n_1, \ldots, n_r) = \max(n_i)$; (iii) $\pi_q(n_1, \ldots, n_r) = 0$; (iv) $\pi_q(n_1, \ldots, n_r) = \max[\pi_s(m_1, \ldots, m_t)$ for $m_i \leqq \sigma_{s_i}(n_1, \ldots, n_r)$; $\pi_{s_i}(n_1, \ldots, n_r)$; $1 \leqq i \leqq t]$, $\sigma_q(n_1, \ldots, n_r) = \max[\sigma_s(m_1, \ldots, m_r)$, $m_i \leqq \sigma_{s_i}(n_1, \ldots, n_r)$; $\sigma_{s_i}(n_1, \ldots, n_r)$, $1 \leqq i \leqq t]$, (v) $\pi_q(0, m) = \pi_s(m)$, $\pi_q(n', m) = \max[\pi_t(n, m, p)$, $p \leqq \sigma_q(n, m)]$; $\sigma_q(0, m) = \sigma_s(m)$, $\sigma_q(n', m) = \max[n', \sigma_q(n, m, p)$, $p \leqq \sigma_t(n, m)]$; $\sigma(n) = \sigma_k(n, 0)$, $\pi(n) = \pi_k(n, 0)$.

The definition is primitive recursive because the simultaneous recursions (v) can be replaced by simple recursions ([8] p. 233 Example 4).

Note that Lemma 1 holds if validity is understood to include validity for incompletely defined predicates.

REMARK 2. While Lemma 1 is stated as a result of informal intuitionistic mathematics in terms of the notion of validity, the argument also establishes

$$A(n, \alpha) \rightarrow (Ep)\mathrm{Prov}[p, \ulcorner(\mathfrak{Z}_{\sigma(n)} \;\&\; \mathfrak{A}_1 \;\&\; \mathfrak{F}^{\alpha}{}_{\pi(n)}) \rightarrow \mathfrak{P}\urcorner].$$

Similarly, using the notation of Remark 1, Lemma 1 specializes to

(5) $$A(n) \rightarrow [(\mathfrak{Z}_{\sigma(n)} \;\&\; \mathfrak{A}_0) \rightarrow \mathfrak{P} \text{ is valid}],$$

and, as above, the proof yields

(6) $$A(n) \rightarrow (Ep)\mathrm{Prov}[p, \ulcorner(\mathfrak{Z} \;\&\; \mathfrak{A}_0) \rightarrow \mathfrak{P}\urcorner].$$

REMARK 3. Incidentally, (6) establishes the completeness of HPC for all formulae $(\mathfrak{Z} \;\&\; \mathfrak{A}_0) \rightarrow \mathfrak{P}$, where $\mathfrak{A}_0$ is obtained from $A(n)$ as in Remark 1. Since the class of valid formulae of this form is non-recursive, *there exists an undecidable fragment of* HPC *for which* HPC *is strongly complete*.

LEMMA 2. *Suppose* $I^*, Z^*, S^*, Q^*, P_0{}^*, P_1{}^*, \ldots, P_k{}^*$ *satisfy the formula*

(7) $$\mathfrak{Z} \;\&\; \mathfrak{A}_1 \;\&\; (x)[Q(x) \vee \neg Q(x)] \;\&\; (x)(y)\neg[Z(y) \;\&\; P_k(x, y)]$$

*in some species* $D$. *Then there is a sequence of elements* $a_n{}^*$ *of* $D$, $n = 0, 1, \ldots,$ *and a function* $\alpha^*$ *such that*: $Z^*(a_0{}^*)$, *and for each* $n$, $S^*(a_n{}^*, a^*{}_{n+1})$; $\alpha^*(n) = 0$

*if* $Q^*(a_n^*)$, $\alpha^*(n) = 1$ *if* $\neg Q^*(a_n^*)$, *and* $(n)\neg A(n, \alpha^*)$. *Also,* (7) *is equivalent to a prenex formula* $\mathfrak{A}$. — *Further, if* $Q^*$ *is completely defined,* $\alpha^*$ *is an effectively defined number-theoretic function; if* $Q^*$ *depends on parameters* $\beta$, $\alpha^*$ *is effective relative to* $\beta$.

Observe first, since $\mathfrak{Z}^*$ holds, for some element $a_0^*$, $Z^*(a_0^*)$, and, since $\mathfrak{Z}^* \to (x)(Ey)_D S^*(x, y)$, there is a function $\nu(n)$ for which $\nu(0) = a_0^*$, $S^*[\nu(n), \nu(n+1)]$; take $\nu(n)$ for $a_n^*$ with $\nu(n)$ possibly $= \nu(m)$ even for $n \neq m$. If a second sequence $b_0^*, b_1^*, \ldots$ satisfies $Z^*(b_0^*)$, $S^*(b_n^*, b^*_{n+1})$ for $n = 0, 1, \ldots$ by the identity axioms in $\mathfrak{Z}$ and $\mathfrak{A}_1$, $Q^*(a_n^*) \leftrightarrow Q^*(b_n^*)$. Also, since $(x)_D[Q^*(x) \vee \neg Q^*(x)]$, $(n)\{Q^*[\nu(n)] \vee \neg Q^*[\nu(n)]\}$, and so $\alpha^*$ is defined uniquely. — Note that both the $a_n^*$ and $Q^*$ may be incompletely defined, e.g. depend on a parameter $\beta$ over free choice sequences. But since we have $Q^*[\nu(n)] \vee \neg Q^*[\nu(n)]$, for each $n$ the decision between $Q^*[\nu(n)]$ and $\neg Q^*[\nu(n)]$ can be made on the basis of a finite number of values of $\beta$, and so $\alpha^*$ is effective in $\beta$.

Consider any $n$; $Z^*$, $S^*$, $Q^*$ satisfy $\mathfrak{F}^{\alpha^*}{}_{\pi(n)}$ by choice of $\alpha^*$ and so, since $\mathfrak{Z}^*$, $\mathfrak{A}_1^*$, by Lemma 1

$$A(n, \alpha^*) \to (Ex)(Ey)_D[Z^*(y) \,\&\, P_k^*(x, y)].$$

By contraposition, and, by [8] p. 162 *86,

$$(x)(y)_D\neg[Z^*(y) \,\&\, P_k^*(x, y)] \to \neg(Ex)(Ey)_D[Z^*(y) \,\&\, P_k^*(x, y)],$$

we get $\neg A(n, \alpha^*)$. Since this is so for each $n$, $(E\alpha)_B(n)\neg A(n, \alpha)$.

Since (7) is a conjunction of prenex formulae it is equivalent in HPC to the prenex formula obtained by using distinct symbols for all the quantifiers of (7) and bringing them forward, [8] p. 162 *89 and *91.

Corollary. *Contraposing Lemma 2 and quantifying,*

$$(\alpha)_B\neg\neg(En)A(n, \alpha) \to (D)(I^*)(Z^*)(S^*)(Q^*)(P_0^*)(P_1^*)\ldots(P_k^*)\neg\mathfrak{A}_D,$$

*since* $\neg(E\alpha)_B(n)\neg A(n, \alpha) \leftrightarrow (\alpha)_B\neg\neg(En)A(n, \alpha)$.

Lemma 3. *For* $\mathfrak{A}$ *of Lemma* 2, Prov$(p, \ulcorner\neg\mathfrak{A}\urcorner) \to (\alpha)_B(En)A(n, \alpha)$.

Observe that (7) has the form $(Ex)Z(x) \,\&\, (x)(Ey)S(x, y) \,\&\, \mathfrak{A}_2$, where $\mathfrak{A}_2$ is a conjunction of purely universal formulae. By [12] Theorem 8 (Herbrand's theorem for negations of prenex formulae of HPC), from a proof of $\neg\mathfrak{A}$ can be obtained a finite number $q$ of substitution instances $\mathfrak{A}_2{}^j$ of $\mathfrak{A}_2$ and terms $\lambda_j$, $1 \leq j \leq q$, such that

(8) $$\neg[\mathfrak{A}_2{}^1 \,\&\, Z(b_1) \,\&\, S(\lambda_1, a_1) \,\&\, \ldots \,\&\, \mathfrak{A}_2{}^q \,\&\, Z(b_q) \,\&\, S(\lambda_q, a_q)]$$

is provable in the (intuitionistic) propositional calculus, where the only term occurring in $\mathfrak{A}_2{}^1$ is $a_0$, $\lambda_1$ denotes $a_0$, and, for each $j \leq q$, $\lambda_j$ denotes one of the terms $a_i$, $b_i$, with $i < j$. Only individual variables occur because $\mathfrak{A}$ does not contain function symbols. Now interpret the symbols of $\mathfrak{A}$ in the sense which led to the axioms $\mathfrak{Z}$ and $\mathfrak{A}_1$, namely: $a_0, b_1, \ldots, b_q$

all denote zero; $a_1$ denotes unity; if $\lambda_j$ denotes the number $\bar{\lambda}_j$, then $a_j$ denotes the successor of $\bar{\lambda}_j$, $Z(x)$ denotes $\bar{x} = 0$, $S(x, y)$ denotes $\bar{y} = \bar{x}+1$, $I(x, y)$ denotes $\bar{x} = \bar{y}$, $Q(x)$ denotes $\alpha(\bar{x}) = 0$, $P_0(x, y)$ denotes $\alpha(\bar{x}) = \bar{y}$, $P_q(x_1, \ldots, x_r, y)$ denotes $f_q(\bar{x}_1, \ldots, \bar{x}_r) = \bar{y}$.

Then the formula (8) reduces to

$$(9) \qquad \neg\{\neg[P_k(x^1, y^1) \,\&\, Z(y^1)] \,\&\, \ldots \,\&\, \neg[P_k(x^q, y^q) \,\&\, Z(y^q)]\},$$

since all the other formulae in the square bracket of (8) reduce to true statements. The interpretation of (9) is

$$\neg[f_k(\bar{x}^1, \alpha) \neq 0 \,\&\, \ldots \,\&\, f_k(\bar{x}^q, \alpha) \neq 0]$$

where $\bar{x}^j \leqq q$ for all $j$. Since $f_k(n, \alpha) = 0 \vee f_k(n, \alpha) \neq 0$,

$$(En)[n \leqq q \,\&\, f_k(n, \alpha) = 0].$$

Since this is so for each $\alpha$, $(\alpha)_B(En)A(n, \alpha)$.

Remark 4. Note, as is to be expected from the fan theorem, the least $n$ satisfying $f_k(n, \alpha) = 0$ depends only on the first $\pi(q)$ values of $\alpha$, where $\pi$ was defined in Lemma 1.

Theorem 1 follows by choosing for $\mathfrak{A}$ the formula of Lemma 2, and applying Lemma 2 and Lemma 3.

7. Theorem 2. *For any primitive recursive $A(n)$, there is a negative formula $\mathfrak{B}$ of* HPC *such that completeness of* HPC *in sense* (1), *and hence also in sense* (1′), *for $\neg\mathfrak{B}$ implies*

$$\neg\neg(En)A(n) \rightarrow (En)A(n).$$

Proof. We take for $\mathfrak{B}$ the formula $(\mathfrak{Z} \,\&\, \mathfrak{A}_0 \,\&\, \neg\mathfrak{P})^{-0}$, where, for any formula $\mathfrak{C}$ of HPC, $\mathfrak{C}^-$ is obtained by replacing each prime part $E$ of $\mathfrak{C}$ by $\neg\neg E$, and $\mathfrak{C}^{-0}$ from $\mathfrak{C}^-$ by putting $\neg(x)\neg A(x)$ for $(Ex)A(x)$ and $\neg(\neg A \,\&\, \neg B)$ for $A \vee B$.[8]

Lemma 4. $(\mathfrak{Z} \,\&\, \mathfrak{A}_0 \,\&\, \neg\mathfrak{P})^{-0} \rightarrow \neg[(\mathfrak{Z}^-{}_{\sigma(n)} \,\&\, \mathfrak{A}_0{}^-) \rightarrow \mathfrak{P}^-]$, *for each n.*

For any quantifier-free $\mathfrak{C}$ without disjunction symbols,[9] by Lemma 43b of [8] p. 496, $\neg\neg\mathfrak{C}^- \leftrightarrow \mathfrak{C}^-$, $\neg\neg\mathfrak{C}^{-0} \leftrightarrow \mathfrak{C}^{-0}$, and by Theorem 59 of [8] p. 492, since $\mathfrak{C}$, $\mathfrak{C}^-$ and $\mathfrak{C}^{-0}$ are classically equivalent, $\neg\neg\mathfrak{C} \leftrightarrow \neg\neg\mathfrak{C}^- \leftrightarrow \neg\neg\mathfrak{C}^{-0}$, and so $\mathfrak{C}^{-0} \leftrightarrow \mathfrak{C}^-$. By quantification, for purely universal $\mathfrak{C}$ without disjunction symbols, $\mathfrak{C}^{-0} \leftrightarrow \mathfrak{C}^-$; in particular $\mathfrak{A}_0{}^{-0} \rightarrow \mathfrak{A}_0{}^-$, $\neg\mathfrak{P}^{-0} \leftrightarrow \neg\mathfrak{P}^-$, $\mathfrak{U}^{-0} \leftrightarrow \mathfrak{U}^-$.

Finally we show that for each natural number $q$, $\mathfrak{S}^{-0} \rightarrow \neg\neg\mathfrak{S}_q{}^-$. Denote $Z(x) \,\&\, S(x, x_1) \,\&\, \ldots \,\&\, S(x_{q-1}, x_q)$ by $S_q$ and $(x)(x_1)\ldots(x_q)\neg S_q$ by $\mathfrak{S}_q$.

[8] $\mathfrak{C}^{-0}$ is $\mathfrak{C}^{\dagger\circ}$ in the notation of [8] p. 494. Since one introduces negation symbols in forming $\mathfrak{C}^-$, we prefer our notation. — No disjunction symbols occur in the formulae $\mathfrak{C}$ considered below.

[9] $\mathfrak{C} \rightarrow \mathfrak{C}^-$ is not correct for *all* quantifier-free $\mathfrak{C}$; e.g. not if $\mathfrak{C}$ is $(A \vee B) \rightarrow C$.

Since $\neg(\neg S_q \,\&\, S_q)$, by quantification $\neg[\mathfrak{S}_q \,\&\, S_{q-1} \,\&\, S(x_{q-1}, x_q)]$, with only one occurrence of $x_q$. Hence $\neg(\mathfrak{S}_q \,\&\, S_{q-1} \,\&\, (Ex_q)S(x_{q-1}, x_q)]$ and, by Theorem 59 of [8], $\neg[\mathfrak{S}_q \,\&\, S_{q-1} \,\&\, \neg\neg(Ex_q)S(x_{q-1}, x_q)]$. By [8] p. 162 *86,

$$(10) \qquad \neg\neg(Ex)A(x) \leftrightarrow \neg(x)\neg A(x),$$

so $\neg\neg(Ex_q)S(x_{q-1}, x_q) \leftrightarrow \neg(x_q)\neg S(x_{q-1}, x_q)$ and quantifying, $\neg[\mathfrak{S}_q \,\&\, S_{q-1} \,\&\, (x_{q-1})\neg(x_q)\neg S(x_{q-1}, x_q)]$, i.e. $\neg[\mathfrak{S}_q \,\&\, S_{q-1} \,\&\, (x)\neg(y)\neg S(x, y)]$. Repeating the argument and contracting pleonasms $\neg[\mathfrak{S}_q \,\&\, \neg(x)\neg Z(x) \,\&\, (x)\neg(y)\neg S(x,y)]$; hence $[\neg(x)\neg Z(x) \,\&\, (x)\neg(y)\neg S(x, y)] \rightarrow \neg\mathfrak{S}_q$. Since by (10), $\neg\mathfrak{S}_q \leftrightarrow \neg\neg\mathfrak{E}_q^-$, we get $\mathfrak{Z}^{-0} \rightarrow \neg\neg\mathfrak{Z}_q^-$ as required.

Since $(\mathfrak{Z} \,\&\, \mathfrak{A}_0 \,\&\, \neg\mathfrak{P})^{-0} \rightarrow (\neg\neg\mathfrak{Z}^-{}_{\sigma(n)} \,\&\, \neg\neg\mathfrak{A}_0^- \,\&\, \neg\mathfrak{P}^-)$, by $(\neg\neg A \,\&\, \neg\neg B \,\&\, \neg C) \rightarrow \neg[(A \,\&\, B) \rightarrow C]$, *60d of p. 119 of [8], the lemma is proved.

Corollary. $(\mathfrak{Z}^* \,\&\, \mathfrak{A}_0^* \,\&\, \neg\mathfrak{P}^*)_D{}^{-0} \rightarrow (n)\neg A(n)$.

By (5), $A(n) \rightarrow [(\mathfrak{Z}_{\sigma(n)} \,\&\, \mathfrak{A}_0) \rightarrow \mathfrak{P}$ is valid]. Restricting the species considered in the definition of validity to those which are double negations,

$$A(n) \rightarrow \{(\mathfrak{Z}^-{}_{\sigma(n)} \,\&\, \mathfrak{A}_0^-) \rightarrow \mathfrak{P}^- \text{ is valid}\}.$$

Contraposing, and quantifying over $n$, the corollary follows. Here, as in the Remark below, validity (correctness) includes validity for incompletely defined species.

Remark 5. Lemma 4, and the construction of $\sigma(n)$, can be avoided by a detour via CPC, as suggested by Kleene. — For any formula $\mathfrak{C}$ in the notation of predicate logic, $\vdash_{\text{HPC}} \mathfrak{C} \rightarrow \vdash_{\text{HPC}} \mathfrak{C}^{-0}$ since (i) $\vdash_{\text{HPC}} \mathfrak{C} \rightarrow \vdash_{\text{CPC}} \mathfrak{C}$, HPC being a subsystem of CPC, (ii) $\vdash_{\text{CPC}} \mathfrak{C} \rightarrow \vdash_{\text{CPC}} \mathfrak{C}^{-0}$ since $\mathfrak{C}$ and $\mathfrak{C}^{-0}$ are classically equivalent, (iii) $\vdash_{\text{CPC}} \mathfrak{C}^{-0} \rightarrow \vdash_{\text{HPC}} \mathfrak{C}^{-0}$ by [8] Theorem 60 (d) p. 495. Further, by the correctness of HPC, $\neg\mathfrak{B}^* \rightarrow (p)\neg\text{Prov}(p, \ulcorner\mathfrak{B}\urcorner)$, and, so, since $\mathfrak{B}$ is $[(\mathfrak{Z} \,\&\, \mathfrak{A}_0) \rightarrow \mathfrak{P}]^{-0}$, $\neg\mathfrak{B}^* \rightarrow (p)\neg\text{Prov}[p, \ulcorner(\mathfrak{Z} \,\&\, \mathfrak{A}_0) \rightarrow \mathfrak{P}\urcorner]$. Contraposing (6) we get the corollary to Lemma 4. (Cf. Remark 6 below.)

Lemma 5. *For prenex* $\mathfrak{C}$ *of* HPC, $\mathfrak{C} \rightarrow \mathfrak{C}^{-0}$ *and so* $\neg\mathfrak{C}^{-0} \rightarrow \neg\mathfrak{C}$.

As in Lemma 4, for quantifier-free $\mathfrak{C}$, $\neg\mathfrak{C}^{-0} \leftrightarrow \neg\mathfrak{C}$, and, since $\mathfrak{C} \rightarrow \neg\neg\mathfrak{C}$, for such $\mathfrak{C}$, $\mathfrak{C} \rightarrow \mathfrak{C}^{-0}$. Since $(Ex)A(x) \rightarrow \neg(x)\neg A(x)$, the lemma follows for all prenex $\mathfrak{C}$ by quantification.

(Note that $\mathfrak{C} \rightarrow \mathfrak{C}^{-0}$ does not hold generally, e.g. if $\mathfrak{C}$ is the formula of Footnote 2.)

To prove Theorem 2, assume $\neg\neg(En)A(n)$. Contraposing the corollary to Lemma 4, $\neg\mathfrak{B}$ is valid (for arbitrary species). So by the assumed completeness of HPC for $\neg\mathfrak{B}$ in sense (1), $\vdash_{\text{HPC}} \neg\mathfrak{B}$. Now apply Lemma 5 to $(\mathfrak{Z} \,\&\, \mathfrak{A}_0 \,\&\, \neg\mathfrak{P})$ as $\mathfrak{C}$, so that $\vdash_{\text{HPC}} \neg\mathfrak{B} \rightarrow \vdash_{\text{HPC}} \neg\mathfrak{C}$. By specialization of Lemma 3 (with $\alpha$ absent), $(En)A(n)$.

8. Remark 6. Evidently, Theorem 1 cannot be strengthened by taking $\mathfrak{A}$ negative, since, by [12] Theorem 4 the weak completeness of the negative

fragment can be proved by the usual methods while, as shown below, (3) cannot. — It may be noted that, since $\mathfrak{A}$ (of Theorem 1 above) is prenex, $\vdash_{\mathrm{HPC}} \neg\mathfrak{A} \leftrightarrow \vdash_{\mathrm{HPC}} \neg\mathfrak{A}^{-0}$ (by Theorem 8 of [12]). But, in the absence of a completeness proof, this equivalence with respect to derivability in HPC does not ensure equivalence with respect to validity. The assumption of completeness of HPC for $\neg\mathfrak{A}$ is *stronger* than the assumption of completeness for $\neg\mathfrak{A}^{-0}$: since we do not have $\neg\mathfrak{A} \to \neg\mathfrak{A}^{-0}$ (in contrast to Lemma 5), $\neg\mathfrak{A}$ may be valid even if $\neg\mathfrak{A}^{-0}$ is not. In particular, while $(x)_D[Q^*(x) \vee \neg Q^*(x)]$ and $(x)(Ey)_D S^*(x, y)$ are both used essentially in Lemma 2, their negative versions are inadequate: $(x)_D\neg(y)_D\neg S^*(x, y)$ does not ensure $\neg\neg(E\nu^*)(n)\neg\neg S^*[\nu(n), \nu(n+1)]$; and even if we have such a $\nu(n)$, since $(x)_D[Q^*(x) \vee \neg Q^*(x)]^{-0}$ is identically valid, $Q^*[\nu(n)] \vee \neg Q^*[\nu(n)]$ need not hold so that $\alpha^*$ is not completely defined. — One difference between Lemma 4 and Kleene's more elegant proof in Remark 5 is that Lemma 4 uses only implications between formulae of HPC while Remark 5 appeals to the *formal* derivability result $\vdash_{\mathrm{HPC}} \mathfrak{C} \to \vdash_{\mathrm{HPC}} \mathfrak{C}^{-0}$, which does not establish a clear connection between the validity of $\mathfrak{C}$ and of $\mathfrak{C}^{-0}$. —

**9. Higher-order systems.** The formal non-derivability results of the present paper are obtained by use of the interpretation of [11] para. 3.52 of intuitionistic systems in terms of functionals of finite type. This auxiliary apparatus is described in the present section.

NOTATION OF ARITHMETIC OF FINITE TYPE ($\mathrm{HA}^\omega$). (i) Inductive definition of *type*: (0) is a type; if $\tau_1, \ldots, \tau_k$ and $\tau$ are types, so is $(\tau; \tau_1, \ldots, \tau_k)$. The *finite* types are those obtained by a finite number of applications of the preceding two clauses.

(ii) The *symbols* used are (a) the logical constants, the individual constant 0 of type (0), the (successor) symbol ′ of type ((0); (0)), an infinite list of (constants) $\varphi_n{}^\tau$ of each type $\tau$, (b) an infinite list of bound and free variables of each type, (c) the relation symbol =.

(iii) *Terms* are formed according to the rules of the simple theory of types with $(\tau; \tau_1, \ldots, \tau_k)$ representing the type of a function(al) with arguments of type $\tau_1, \ldots, \tau_k$ and values of type $\tau$: 0 is a term of type (0); $n, n_1, \ldots, m, \ldots$ are variables of type (0); for other types $\tau$, $x_i{}^\tau$, $y_i{}^\tau$ are variables of type $\tau$; if $\mathfrak{t}$ is of type (0), so is $\mathfrak{t}'$. If $\mathfrak{t}$ is a term of type $(\tau; \tau_1, \ldots, \tau_k)$ and not the symbol′, and $\mathfrak{t}_i$ is of type $\tau_i$, then $\mathfrak{t}(\mathfrak{t}_1, \ldots, \mathfrak{t}_k)$ is of type $\tau$; if $\mathfrak{t}$ with the free variables $x_i{}^{\tau_i}$ is a term, so is the expression obtained by *substituting* the terms $\mathfrak{t}_i$ for $x_i{}^{\tau_i}$ (though the intended operation is clear, an explicit formulation is lengthy). — We shall use Greek lower case letters $\alpha, \beta, \ldots$ for variables of type ((0); (0)), and Greek capitals $\Phi$ for variables of type ((0); ((0); (0))).

(iv) If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are terms of the same type, $\mathfrak{t}_1 = \mathfrak{t}_2$ is a prime *formula*, and composite formulae are obtained by the usual logical operations.

INTERPRETATION OF $HA^\omega$. One familiar intuitionistic interpretation of the quantifier-free fragment of $HA^\omega$ is given in Gödel's [4] p. 282–283, where the terms of type (0) are interpreted as denoting natural numbers, terms of other types as *effective functions*, 0 as zero, ′ as the successor, = as identity, i.e. equality, for type (0), and intensional identity between non-zero types [4] p. 283 Footnote 3. The logical symbols of the fragment are interpreted in the two-valued sense,[10] and free variable formulae are interpreted as assertions stating that a constructive proof is available which shows that every instance of the free variable formula holds.[11]

INTERPRETATION OF $HA^\omega$ (continued). For our purposes, different interpretations will be needed. First, the ranges of the variables of non-zero type are restricted; specifically, not *all* effective functions and functionals (defined on effective functions) are used, but only those generated by certain *schemata*, the schemata used depending on the particular formal system of axioms and rules considered: in general, the functionals obtained by the schemata are defined not only for effective function(al)s as arguments, but also for free choice sequences. Thus if $\mathfrak{F}^\tau$ denotes the class of functionals generated by our schemata, the functionals of $\mathfrak{F}(\tau; \tau_1, \ldots, \tau_k)$ are defined for a *larger* class of arguments than $\mathfrak{F}^{\tau_1} \times \ldots \times \mathfrak{F}^{\tau_k}$. — Finally, since we use quantified formulae, the logical constants are interpreted intuitionistically.

REMARK 7. The non-derivability results below depend on the fact that the theorems of the formal systems considered are valid under the (restricted) interpretations of the kind described.

**10.** REINTERPRETATION OF $HA^\omega$ (GENERALIZED REALIZABILITY). With each formula $\mathfrak{A}$ of $HA^\omega$ we associate a formula $\mathfrak{A}^*$, also of $HA^\omega$, of the form $(E\mathfrak{x})\mathscr{A}(\mathfrak{x})$ where $(E\mathfrak{x})$ denotes a string of existential quantifiers and $\mathscr{A}(\mathfrak{x})$ contains neither existence nor disjunction symbols. $\mathfrak{A}^*$ has the same free variables as $\mathfrak{A}$.

(i) If $\mathfrak{A}$ does not contain existence or disjunction symbols, then $\mathfrak{A}^*$ is $\mathfrak{A}$ itself, and so $\mathscr{A}(\mathfrak{x})$ (denotes a formula which) does not contain $\mathfrak{x}$.

(ii) If $\mathfrak{A}$ is $\mathfrak{A}_1 \,\&\, \mathfrak{A}_2$, then $\mathfrak{A}^*$ is $(E\mathfrak{x}_1)(E\mathfrak{x}_2)[\mathscr{A}_1(\mathfrak{x}_1) \,\&\, \mathscr{A}_2(\mathfrak{x}_2)]$.[12]

[10] [4] p. 284 line 4; in the quantifier-free fragment this is equivalent to the intuitionistic interpretation because (intensional) equality is decidable. — For Gödel's purpose the introduction of the intuitionistic logical constants would have been nugatory because his main aim was to show that the latter could be eliminated in favour of effective functions of finite type, within the context considered.

[11] Strictly speaking, reference to the range of a variable is only a *façon de parler* in intuitionistic mathematics. What is required are conditions stating which constructions are proofs of a statement containing variables.

[12] More explicitly, $\mathfrak{A}_1^*$ is $(E\mathfrak{x}_1)\mathscr{A}_1(\mathfrak{x}_1)$ and $\mathfrak{A}_2^*$ is $(E\mathfrak{x}_2)\mathscr{A}_2(\mathfrak{x}_2)$ (if necessary the variables in $\mathfrak{x}_1$ and in $\mathfrak{x}_2$ are renamed as $\mathfrak{x}_1'$ and $\mathfrak{x}_2'$ so as to avoid clashing of variables), $(E\mathfrak{x})$ denotes $(E\mathfrak{x}_1')(E\mathfrak{x}_2')$, and $\mathscr{A}(\mathfrak{x})$ denotes $\mathscr{A}_1(\mathfrak{x}_1') \,\&\, \mathscr{A}_2(\mathfrak{x}_2')$. So, if $\mathscr{A}_1$, $\mathscr{A}_2$ do not contain disjunction or existential symbols, neither does $\mathscr{A}$.

(iii) If $\mathfrak{A}$ is $\mathfrak{A}_1 \vee \mathfrak{A}_2$, then $\mathfrak{A}^*$ is $(En)(E\mathfrak{x}_1)(E\mathfrak{x}_2)\{[n=0 \rightarrow \mathscr{A}_1(\mathfrak{x}_1)]\ \&\ [n\neq 0 \rightarrow \mathscr{A}_2(\mathfrak{x}_2)]\}$.

(iv) If $\mathfrak{A}$ is $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ then $\mathfrak{A}^*$ is $(E\mathfrak{x})(\mathfrak{y})\{\mathscr{A}_1(\mathfrak{y}) \rightarrow \mathscr{A}_2[\mathfrak{x}(\mathfrak{y})]\}$, where the type of $\mathfrak{x}$ is determined as follows: if the types of the sequence $\mathfrak{x}_1$ are $\tau_1{}^1, \ldots, \tau_p{}^1$ and of $\mathfrak{x}_2$ are $\tau_1{}^2, \ldots, \tau_q{}^2$, then the types of the sequence $\mathfrak{y}$ are those of $\mathfrak{x}_1$ and of the sequence $\mathfrak{x}$ are $(\tau_i{}^2; \tau_1{}^1, \ldots, \tau_p{}^1)$ for $i = 1, 2, \ldots, q$.

(v) If $\mathfrak{A}$ is $\neg\mathfrak{A}_1$, then $\mathfrak{A}^*$ is $(\mathfrak{y})\neg\mathscr{A}_1(\mathfrak{y})$, and so $\mathfrak{x}$ does not occur in $\mathfrak{A}^*$ at all.

(vi) If $\mathfrak{A}$ is $(\mathfrak{y})\mathfrak{A}_1(\mathfrak{y})$ and $[\mathfrak{A}_1(\mathfrak{y})]^*$ is $(E\mathfrak{x}_1)\mathscr{A}_1(\mathfrak{y}, \mathfrak{x}_1)$, then $\mathfrak{A}^*$ is $(E\mathfrak{x})(\mathfrak{y})\mathscr{A}_1[\mathfrak{y}, \mathfrak{x}(\mathfrak{y})]$, the types of the sequence $\mathfrak{x}$ being chosen as under (iv).

(vii) If $\mathfrak{A}$ is $(E\mathfrak{x}_1)\mathfrak{A}_1(\mathfrak{x}_1)$ and $[\mathfrak{A}_1(\mathfrak{x}_1)]^*$ is $(E\mathfrak{x}_2)\mathscr{A}_1(\mathfrak{x}_1, \mathfrak{x}_2)$, then $\mathfrak{A}^*$ is $(E\mathfrak{x}_1)(E\mathfrak{x}_2)\mathscr{A}_1(\mathfrak{x}_1, \mathfrak{x}_2)$, and $(E\mathfrak{x})$ denotes $(E\mathfrak{x}_1)(E\mathfrak{x}_2)$.

REMARK 8. If $\mathfrak{A}$ does not contain implication symbols, $\mathfrak{A} \leftrightarrow \mathfrak{A}^*$ on the intended interpretations of $HA^\omega$, provided only the ranges of the variables of higher types considered are large enough to permit (vi). On the other hand, if $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ are arbitrary formulae (of $HA^\omega$) and $\mathfrak{A}$ is $[\neg\mathfrak{A}_1 \rightarrow (\mathfrak{A}_2 \vee \mathfrak{A}_3)] \rightarrow [(\neg\mathfrak{A}_1 \rightarrow \mathfrak{A}_2) \vee (\neg\mathfrak{A}_1 \rightarrow \mathfrak{A}_3)]$, then $\mathfrak{A}^*$ is valid, but in general $\mathfrak{A}$ is not.

RELATION TO KLEENE'S RECURSIVE REALIZABILITY. For each $\mathfrak{A}$ whose free variables are $x_i^{\tau_i}$, $1 \leqq i \leqq k$, we define the relation $R_{\mathfrak{A}}(\mathfrak{x}; x_1^{\tau_1}, \ldots, x_k^{\tau_k})$ (read: the sequence of functionals $\mathfrak{x}$ *realizes* $\mathfrak{A}$ for the arguments $x_1^{\tau_1}, \ldots, x_k^{\tau_k}$) by induction with respect to the logical structure of $\mathfrak{A}$ as in [8] pp. 502–503. — In rewriting Kleene's definition, note (i) that we use strings of variables as in [4], while [8] contracts them; (ii) we use functional variables instead of Gödel *numbers* of partial recursive functions in [8] (this is not essential if the functional variables are restricted to range over *effective operations* of [11] p. 117 4.2 without the extensionality condition). — A difference between [8] and the above is that in (iv) we require $\mathfrak{x}$ to be a functional defined on all the arguments of our range, while [8] p. 503 Clause 4 only requires the corresponding $\mathfrak{x}$ to be defined for those $\mathfrak{y}$ which satisfy $\mathscr{A}_1(\mathfrak{y})$. But, for our purposes the essential difference is that we envisage interpretations where the ranges of the variables $\mathfrak{x}$ do not consist of *all* the effective functions in the sense of [4], when *effective* and *recursive* are identified.

**11.** We note here the following result which is needed below.

THEOREM 3. *There is a primitive recursive relation $A_k(n, \alpha)$ such that*

$$[(\alpha)\neg\neg(En)A_k(n, \alpha) \rightarrow \neg\neg(\alpha)(En)A_k(n, \alpha)]^* \tag{11}$$

*is intuitionistically refutable if, under the realizability interpretation considered, the range of the function variables $\alpha$ considered is a subspecies of the species of recursive functions, and each functional of type $((0); ((0); (0)))$ in the range considered is defined for all free choice sequences.*

For, (11) is $(\alpha)\neg(n)\neg A_k(n, \alpha) \rightarrow \neg(\Phi)\neg(\alpha)A_k[\Phi(\alpha), \alpha]$. Consider now Kleene's $A_k$ of [7] such that $(\alpha)_{\text{Rec}}(En)A_k(n, \alpha)$ is provable intuitionistically, and, by means of the fan theorem $\neg(\alpha)(En)A_k(n, \alpha)$, where $\alpha$ ranges over all free choice sequences. Thus, for our range of $\Phi$, $\neg(E\Phi)(\alpha)A_k[\Phi(\alpha), \alpha]$, i.e. $(\Phi)\neg(\alpha)A_k[\Phi(\alpha), \alpha]$. This refutes (11).

The following result is needed for a side result below. (The proof merely verifies intuitionistically that to any recursive sequence of recursive functions there is a recursive function not included in the sequence.)

THEOREM 4. *For every recursive enumeration* $U[\mu_y T_2(e, r, m, y)]$ *in the notation of* [8] *satisfying* $(r)(m)(Ey)T_2(e, r, m, y)$, *of a sequence of (recursive) functions* $\varphi_0(m), \ldots, \varphi_r(m), \ldots$, *there is a primitive recursive* $B_e$ *such that*

$$(12) \qquad [(n)\neg\neg(Em)B_e(n, m) \rightarrow \neg\neg(n)(Em)B_e(n, m)]^*$$

*is refutable if, in the generalized realizability interpretation considered, the range of the type* ((0); (0)) *variables is included in the sequence* $\varphi_0, \varphi_1, \ldots$.

By a diagonal construction, we choose for $B_e(n, m)$ a formula such that $(n)(Em)B_e(n, m)$, but, for each $r$,

$$\neg(n)B_e\{n, U[\mu_y T_2(e, n, r, y)]\},$$

e.g. $B_e(n, m) \leftrightarrow (Ey)[y<m \;\&\; U(m)=U(y)+1 \;\&\; T_2(e, n, n, y)]$.

For, (12) is $(n)\neg(m)\neg B_e(n, m) \rightarrow \neg(\alpha)\neg(n)B_e[n, \alpha(n)]$.

Now, on the assumption $(r)(m)(Ey)T_2(e, r, m, y)$, we evidently have $(n)(Em)B_e(n, m)$ and hence $(n)\neg(m)\neg B_e(n, m)$. On the other hand $(r)\neg(n)B_e[n, \varphi_r(n)]$, because, for the unique $y$ which satisfies $T_2(e, r, r, y)$, $\varphi_r(r) = U(y)$ and so $\neg B_e[r, \varphi_r(r)]$ since $U(p) \leqq p$ for all $p$.

Note that the formula $A_k$ in Theorem 3 is chosen independently of the range of the ((0); (0)) variables, as long as they range only over recursive functions. In Theorem 4, $B_e$ depends on the particular recursive enumeration.

**12.** Before Theorems 3 or 4 can be used for formal non-derivability results, it is necessary to verify that generalized realizability (for ranges of variables which satisfy the conditions of Theorems 3 and 4) applies to the systems considered. For instance, this is not quite clear for Kleene's system [9], as it stands. The axiom of choice $(x)(Ey)\mathfrak{A}(x,y) \rightarrow (E\alpha)(x)\mathfrak{A}[x,\alpha(x)]$, stated in [9], p. 286 (3), asserts the existence of a function $\alpha$, and it is not clear (for Theorem 4) that such an $\alpha$ can always be found in a recursive sequence of functions uniformly for all formulae $\mathfrak{A}$. Similarly, the fan theorem [9] p. 286 (4) asserts the existence of a bound $z$ depending on $\beta$, which is not immediately characterizable for arbitrary predicates $\mathfrak{A}$. In fact, for non-recursive $\mathfrak{A}$, the formula (4) on p. 286 of [9] is generally false when the logical constants are interpreted set-theoretically; only on the intuitionistic interpretation, which asserts the existence of $z$ on the basis of a *proof* of the premise, is the consistency of Kleene's system immediately evident (cf. Remark 11 below). In any case, without further analysis the exact dependence of $z$ on $\beta$ is not apparent.

THE FORMAL SYSTEM $\mathrm{HA_{NF}}^{\omega}$ (Negative fragment of intuitionistic arithmetic of finite order with the fan theorem). We use the notation of $\mathrm{HA}^{\omega}$ *without* existence or disjunction symbols.

(a) *Logical axioms and rules*: the intuitionistic predicate calculus for negative formulae of [6] pp. 350–352 extended to finite types, by means of the schemata

$$(x^{\tau})\mathfrak{A}(x^{\tau}) \rightarrow \mathfrak{A}(\mathfrak{t}), \qquad \frac{\mathfrak{B} \rightarrow \mathfrak{A}(x^{\tau})}{\mathfrak{B} \rightarrow (x^{\tau})\mathfrak{A}(x^{\tau})}$$

where $\mathfrak{t}$ is any term of type $\tau$, and $x^{\tau}$ does not occur in $\mathfrak{B}$; also identity axioms. (Note that, as in [4], we do not include axioms of extensionality.)

(b) *Non-logical axioms and rules*: $n' \neq 0$, $n' = m' \rightarrow n = m$,

$$\frac{\mathfrak{A}(0),\ \mathfrak{A}(n) \rightarrow \mathfrak{A}(n')}{\mathfrak{A}(n)}$$

applied to formulae $\mathfrak{A}$ of the present fragment. For

$$\tau_0 = ((0);((0);((0);(0))),((0);(0))),$$

and free variables $\alpha$, $\beta$, $\gamma$,

(13) $$(n)[\beta(n) \leqq \alpha(n) \;\&\; \gamma(n) \leqq \alpha(n)] \;\&\; (n)[n < \varphi_0{}^{\tau_0}(\Phi, \alpha) \rightarrow \beta(n) = \gamma(n)]\} \rightarrow \Phi(\beta) = \Phi(\gamma),$$

where (13) asserts the existence of a modulus of uniform continuity of $\Phi$ on the set of free choice sequences bounded by $\alpha$. Finally, we have a sequence of equations as additional axioms (cf. [4] p. 283 para. 2), for $p = 1, 2, \ldots$ (with $\tau, \tau_1, \ldots, \tau_t$ depending on $p$)

$$\varphi^{\bar{\tau}}{}_{2p-1}(x_1{}^{\tau_1}, \ldots, x_t{}^{\tau_t}) = \mathfrak{t},\ \bar{\tau} = (\tau; \tau_1, \ldots, \tau_t),$$

where $\mathfrak{t}$ is built up from among $\varphi_k$, $k < 2p-1$, $0$, $'$, and the variables $x_1{}^{\tau_1}, \ldots, x_t{}^{\tau_t}$, and

$$\varphi^{\bar{\tau}}{}_{2p}(0; x_2{}^{\tau_2}, \ldots, x_t{}^{\tau_t}) = \varphi_i{}^{\bar{\tau}'}(x_2{}^{\tau_2}, \ldots, x_t{}^{\tau_t}),\ i < 2p,$$

$$\varphi^{\bar{\tau}}{}_{2p}(n', x_2{}^{\tau_2}, \ldots, x_t{}^{\tau_t}) = \varphi_j{}^{\bar{\tau}''}[n, x_2{}^{\tau_2}, \ldots, x_t{}^{\tau_t}, \varphi^{\bar{\tau}}{}_{2p}(n, x_2{}^{\tau_2}, \ldots, x_t{}^{\tau_t})],\ j < 2p$$

$$\bar{\tau} = (\tau; (0), \tau_2, \ldots, \tau_t),\ \bar{\tau}' = (\tau; \tau_2, \ldots, \tau_t);\ \bar{\tau}'' = (\tau; (0), \tau_2, \ldots, \tau_t, \tau).$$

The sequence is such that for any term $\mathfrak{t}$ built up from the constants of the sequence there is a corresponding $\varphi_{2p-1}$, and for any pair $\varphi_i$, $\varphi_j$ of types satisfying the conditions above there is a corresponding $\varphi_{2p}$.

THE FORMAL SYSTEM $\mathrm{HA_{NB}}^{\omega}$ (Negative fragment of intuitionistic arithmetic of finite order with the bar theorem).

As $HA_{NF}^{\omega}$, but, instead of (13):

$$(14) \qquad (n)[n \leqq \varphi_0^{\tau_0}(\Phi, \alpha) \to \beta(n) = \alpha(n)] \to \Phi(\alpha) = \Phi(\beta).$$

(14) asserts the existence of a modulus of continuity of $\Phi$ at the argument $\alpha$.

LEMMA 6. *If $\mathfrak{A}$ is provable in Kleene's system* [9], *there is a constant $\varphi$ of $HA_{NF}^{\omega}$ such that $\mathscr{A}(\varphi)$ is provable in $HA_{NF}^{\omega}$.*

Actually, the proof of the lemma applies not only to Kleene's system, but to its extension to higher types, including the axioms of choice

$$(x_1^{\tau})(Ex_2^{\tau'})\mathfrak{A}(x_1, x_2) \to (Ex_2^{(\tau';\tau)})(x_1^{\tau})\mathfrak{A}[x_1, x_2(x_1)].$$

The proof proceeds in the familiar way ([4], [8], [11]) by considering each axiom $\mathfrak{A}$, replacing the predicate symbols $P_i(x)$ by $(E\mathfrak{x})\mathscr{A}_i(x, \mathfrak{x})$, forming $\mathfrak{A}^*$ of the form $(E\mathfrak{x})\mathscr{A}(\mathfrak{x})$ and showing that, for a suitable $\bar{\varphi}$ of $HA_{NF}^{\omega}$, $\mathscr{A}(\bar{\varphi})$ is provable in $HA_{NF}^{\omega}$. The translations of the axioms of choice are simply instances of $p \to p$, and the translation of the fan theorem is derivable from (13). The rules of inference are treated similarly.

REMARK 9. An analogous argument applies to a system of analysis when, instead of the fan theorem, the bar theorem is formulated as in [5] 12.22, in the very weak form (cf. para. 5 above)

$$(\alpha)(En)\mathfrak{A}(\alpha, n) \to (\alpha)(En_1)(En_2)(\beta)\{(m)[m \leqq n_1 \to \alpha(m) = \beta(m)] \to \mathfrak{A}(\beta, n_2)\}.$$

In this case for any theorem $\mathfrak{A}$ of this system, $\mathfrak{A}^*$ is provable in $HA_{NB}^{\omega}$.

LEMMA 7. *Every theorem of $HA_{NF}^{\omega}$ is valid, both classically and intuitionistically, if the individual variables are interpreted as ranging over the natural numbers and the higher-type variables as ranging over those recursively continuous (countable) functionals of* [11] *which are defined by the equations for $\varphi_n$, $n = 0, 1, \ldots$.*

The proof consists in showing that the axioms and formation rules for terms (by composition) are satisfied by these ranges. This is clear for all schemata except (13), since they define recursive functionals for arbitrary functionals as arguments. For (13), using the notation of [11], we define a *representing* function $g(U^{(2)}, U^{(1)})$ as follows.

Let $s$ denote (the number of) the finite sequence $\langle (s)_0, \ldots, (s)_r \rangle$, $\mathrm{lh}(s) = r+1$, and, if $U_i^{(1)}$ denotes $\{\ldots, \langle n_{ij}, p_{ij} \rangle, \ldots\}$, $1 \leqq j \leqq k_i$, let $s \subset U_i^{(1)}$ mean: $(j)[n_{ij} < \mathrm{lh}(s) \;\&\; (s)_{n_{ij}} = p_{ij}]$. We now define:

$$g(U^{(2)}, U^{(1)}) = 0,$$

unless $U^{(1)} = \{\langle 0, p_0 \rangle, \ldots, \langle r, p_r \rangle\}$, $U^{(2)} = \{\ldots, \langle U_i^{(1)}, q_i \rangle, \ldots\}$, $U_i^{(1)} = \{\ldots, \langle n_{ij}, p_{ij} \rangle, \ldots\}$, $1 \leqq j \leqq k_i$ and $(s)[\{\mathrm{lh}(s) = r+1 \;\&\; (h)[h < r+1 \to (s)_h \leqq p_h]\} \to (Ei)(s \subset U_i^{(1)})]$. In this case,

$$g(U^{(2)}, U^{(1)}) = 1 + \mu_{t \leqq r}(s)(s')\{\mathrm{lh}(s) = \mathrm{lh}(s') = r+1 \;\&\; (h)[(s)_h \leqq p_h \;\&\;$$
$$(s')_h \leqq p_h) \;\&\; (j)(j \leqq t \to (s)_j = (s')_j)] \to (Ei)(Ei')(s \subset U_i^{(1)} \;\&\; s' \subset U_{i'}^{(1)} \;\&\; q_i = q_{i'})\}.$$

It is easily verified, by means of the fan theorem, that the functional represented by $g$ satisfies (13).

COROLLARY. *The conditions on the ranges of the variables mentioned in Theorems 3 and 4 are satisfied by* $\mathrm{HA}_{\mathrm{NF}}{}^{\omega}$.

REMARK 10. To apply the results to the case of the bar theorem, one must give up the requirement of extensionality for the functionals considered.

*Added March* 24, 1961. To show that no arbitrary continuous (extensional) functional in the sense of [11] satisfies (14), we put $\alpha_0 = \lambda n\, 0$, $\Phi_0 = \lambda\alpha\, 0$, $m_0 = \varphi_0{}^{\tau_0}(\Phi_0, \alpha_0)$, $\Phi_1(\alpha) = \Phi_0(\alpha)$ unless $(r)[r \leqq m_0 \rightarrow \alpha(r) = 0]$ & $\alpha(m_0+1) > m_1$, when $\Phi_1(\alpha) = 1$. Representing functions $k_0{}^{(2)}$ of $\Phi_0$ and $k_1{}^{(2)}$ of $\Phi_1$ are given by:

(i) $k_0{}^{(2)}(U^{(1)}) = 1$ if $U^{(1)} = \{\ldots, \langle r, 0\rangle, \ldots, \langle s, q\rangle, \ldots, \langle n_i, p_i\rangle, \ldots\}$ for some $s \leqq m_0+1$ and all $r < s$, with $q > 0$ if $s \leqq m_0$; otherwise $k_0{}^{(2)}(U^{(1)}) = 0$.

(ii) $k_1{}^{(2)}(U^{(1)}) = k_0{}^{(2)}(U^{(1)})$ unless $k_0{}^{(2)}(U^{(1)}) = 1$ and $s = m_0+1$ and $q > m_1$, in which case $k_1{}^{(2)}(U^{(1)}) = 2$.

If $\tau_1 = ((0); ((0); (0)))$, consider an arbitrary $\varphi_1{}^{\tau_1}$ and any representing function $h$ of $\varphi_1{}^{\tau_1}$. There must be a $U_0{}^{(2)}$ such that $h(U_0{}^{(2)}) \leqq \varphi_1{}^{\tau_1}(\Phi_0)+1$, where $U_0{}^{(2)} = \{\ldots, \langle U_j{}^{(1)}, 0\rangle, \ldots\}$, $j \leqq J$, and $k_0{}^{(2)} \in U_0{}^{(2)}$. By (i)

$$U_j{}^{(1)} = \{\ldots, \langle r, 0\rangle, \ldots, \langle s_j, q_j\rangle, \ldots, \langle n_{ji}, p_{ji}\rangle, \ldots\}.$$

If $m_1 = \max(q_j)$ $(j \leqq J$ and $s_j = m_0+1)$, then $k_1{}^{(2)} \in U_0{}^{(2)}$, and so $\varphi_1{}^{\tau_1}(\Phi_1) = \varphi_1{}^{\tau_1}(\Phi_0)$. But (14) is not satisfied by $\Phi_1$ and $\alpha_0$ if $\varphi_0{}^{\tau_0}(\Phi_1, \alpha_0) = m_0$ and $\beta = \lambda n\, (m_1+1)\delta(n, m_0+1)$ (Kronecker's $\delta$), and so $\varphi_0{}^{\tau_0}(\Phi_1, \alpha_0) \neq m_0$. Since $\varphi_1{}^{\tau_1}$ is arbitrary, this proves the result.

Note that the argument establishes $\varphi_0{}^{\tau_0}(\Phi, \alpha_0) \notin C(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3)$, for wide classes $\mathfrak{F}$ (in [11] p. 114). [*End of addition.*]

To obtain an analogue to Lemma 7 for the system of analysis with the bar theorem, as formulated in Remark 9, we drop *extensionality* from [11] p. 115. Instead we arrange the formal neighbourhoods for each type in a fixed order $U_r{}^{(k)}$, $r = 0, 1, \ldots$, and merely require that $h^{(k+1)}(\{\langle U_0{}^{(k)}, q_0\rangle, \ldots, \langle U_r{}^{(k)}, q_r\rangle\}) = s+1 \rightarrow h^{(k+1)}(\{\langle U_0{}^{(k)}, q_0\rangle, \ldots, \langle U_r{}^{(k)}, q_r\rangle, \langle U_{r+1}{}^{(k)}, q_{r+1}\rangle\}) = s+1$.

It is then verified that these functionals are closed under composition, Gödel's schemata for primitive recursive functionals, and they also satisfy (14).

As an immediate consequence of the present remark, we have that the bar theorem is not formally derivable from the fan theorem (as formulated above).

REMARK 11. In view of Lemmas 6 and 7, the generalized realizability interpretation has a double significance. On the one hand, the formula $\mathfrak{A}^*$ (of $\mathrm{HA}^{\omega}$) may be interpreted as in para. 9, and so, by Lemma 6, if $\mathfrak{A}$ is formally provable in Kleene's system [9] then both $\mathfrak{A}$ and $\mathfrak{A}^*$ are *intuitionistically* valid; thus $\mathfrak{A}^*$ constitutes an intuitionistic *reinterpretation* of the formula $\mathfrak{A}$. On the other hand, if $\mathfrak{A}$ is formally provable in Kleene's system, in general $\mathfrak{A}$ is not valid under the usual *classical* interpretation of the symbolism $HA^{\omega}$. In contrast to first-order arithmetic, second-order intuitionistic arithmetic is *not* a subsystem of classical second-order arithmetic. In particular, one formal consequence of the fan theorem is $\neg(\alpha)(En)(m)[\alpha(n)=0 \vee \alpha(m)\neq 0]$, which is obviously false if $\alpha$ ranges over any class of functions and the logical constants are interpreted classically. Thus, from the classical point of view, a *consistency proof* for Kleene's system is required, and, as seen from Lemma 7, the (generalized) realizability interpretation helps to provide such a consistency proof.

13. THEOREM 5. *For the primitive recursive relation $A_k(n, \alpha)$ of Theorem 3, the formula $\mathfrak{A}$*

$$(\alpha)_B\neg\neg(En)A_k(n,\alpha) \rightarrow \neg\neg(\alpha)_B(En)A_k(n,\alpha)$$

*is not provable in Kleene's analysis.*

If it were, $\mathfrak{A}^*$ would be provable in $HA_{NF}^{\omega}$ by Lemma 6, and this is not possible by the corollary to Lemma 7 and Theorem 3.

Note that, by Remarks 9 and 10, $\mathfrak{A}$ is not provable by means of the bar theorem either.

CONVENTION (applied in Theorem 6 and Lemma 9 below): Although $HA_{NF}^{\omega}$ does not contain existential symbols, we say (i) that $(En)A(n)$ is formally undecidable in $HA_{NF}^{\omega}$ if neither $(n)\neg A(n)$ nor $A(\bar{m})$ can be proved in $HA_{NF}^{\omega}$ for any numeral $\bar{m}$, and (ii) that $(E\alpha)\mathscr{A}(\alpha)$ is provable in $HA_{NF}^{\omega}$ if there is a term $\varphi_p$ such that $\mathscr{A}(\varphi_p)$ is provable in $HA_{NF}^{\omega}$.

THEOREM 6. *If $A(n)$ is primitive recursive and $(En)A(n)$ is formally undecidable in $HA_{NF}^{\omega}$ then*

$$\neg\neg(En)A(n) \rightarrow (Em)A(m) \tag{15}$$

*is not derivable in Kleene's analysis.*

If it were, by Lemma 6, $\neg(n)\neg A(n) \rightarrow A(\bar{m})$ would be provable in $HA_{NF}^{\omega}$ for some $\bar{m}$, since the generalized realizability interpretation of (15) is $(Em)[\neg(n)\neg A(n) \rightarrow A(m)]$.

Since $A(\bar{m})$ is decidable by computation, we have either (a) $A(\bar{m})$ or (b) $\neg A(\bar{m})$. In case (a), $(Em)A(m)$. In case (b), $\neg\neg(n)\neg A(n)$, i.e. $\neg(Em)A(m)$. But this means that $(Em)A(m)$ is formally decidable in $HA_{NF}^{\omega}$.

COROLLARY. *Since $HA_{NF}^{\omega}$ satisfies the conditions for Gödel's first incompleteness theorem, there is a primitive recursive $A$ for which* (15) *is not derivable in Kleene's analysis.*

REMARK 12. *Note that the result also holds if $A(n)$ contains free parameters. — Theorem 4 shows that there is a primitive recursive $B(n, m)$ for* *which*

$$(n)\neg\neg(Em)B(n, m) \to \neg\neg(n)(Em)B(n, m)$$

*is not provable in Kleene's analysis, since the functions of type* ((0); (0)) *defined by the schemata for* $\varphi_n$ *are recursively enumerable.*

**14. Refinements.** Analysis of Lemma 7 leads to a result which is of interest independently of the main problem of the present paper, namely that $HA_{NF^\omega}$ *is an inessential extension of intuitionistic first-order arithmetic* HA and so, by use of Lemma 6, Kleene's analysis is an inessential extension of HA for a class of formulae described below.

DEFINITION. Let $HA_{EF^\omega}$ be the system obtained by extending $HA_{NF^\omega}$ by means of numerical existence symbols, and the schema

(16) $$(n)\mathfrak{A}[n, \mathfrak{t}(n)] \to (n)(Em)\mathfrak{A}(n, m).$$

LEMMA 8. $HA_{EF^\omega}$ *is an inessential extension of* HA.

SKETCH OF PROOF. Consider a formal proof in $HA_{EF^\omega}$ of a formula $\mathfrak{A}$ of HA. In this finite proof a number of non-logical axioms of $HA_{EF^\omega}$ are used, which are also axioms of $HA_{NF^\omega}$. Consider all the symbols for constant functions and functionals $C$ occurring in these axioms. As in [11] p. 114, we can define in HA *representing functions* for these functionals, including $\varphi_0^{\tau_0}$. Restrict the quantifiers of higher type which occur in the given formal proof in $HA_{EF^\omega}$ to elements of $C$. The resulting sequence of formulae is still a formal proof in $HA_{EF^\omega}$. Now translate each formula of the given sequence in terms of representing functions. Since the set $C$ is finite, the translations can be expressed (in the natural way) by means of formulae of HA, where the final formula $\mathfrak{A}$ is unchanged. It remains to be verified that the translations (which are formulae of HA) are also provable in HA. This is straightforward for instances of the logical axioms and rules of inference. The translations of those axioms for $\varphi_n$ which do not contain $\varphi_0^{\tau_0}$ are easily proved by means of the recursion theorem; those containing $\varphi_0^{\tau_0}$ require an analysis of nested occurrences of $\varphi_0^{\tau_0}$ in the symbols which are substituted for $\alpha$, $\beta$, $\gamma$, $\Phi$ in the substitution instances of (13) that appear in the given proof.

LEMMA 9. *If the formula* $\mathfrak{A}$ *of* HA *is* (a) *negative or* (b) *prenex, and* $\mathfrak{A}^*$ *is provable in* $HA_{NF^\omega}$, *then* $\mathfrak{A}$ *is provable in* HA.

Case (a) follows directly from Lemma 8.

For case (b), if $\mathfrak{A}$ is

$$(n_1)(Em_1)\ldots(n_k)(Em_k)A(n_1, \ldots, n_k, m_1, \ldots, m_k),$$

$\mathfrak{A}^*$ is $(E\alpha_1, \ldots, \alpha_k)(n_1)\ldots(n_k)A[n_1, \ldots, n_k, \alpha_1(n_1), \ldots, \alpha_k(n_1, \ldots, n_k)]$, and, by our convention above, $\mathfrak{A}^*$ is provable in $HA_{NF^\omega}$, if, for some $\varphi_{p_1}, \ldots, \varphi_{p_k}$,

$$(n_1)\ldots(n_k)A[n_1, \ldots, n_k, \varphi_{p_1}(n_1), \ldots, \varphi_{p_k}(n_1, \ldots, n_k)]$$

is provable in $\mathrm{HA}_{\mathrm{NF}^\omega}$. By (16), $\mathfrak{A}$ is provable in $\mathrm{HA}_{\mathrm{EF}^\omega}$, and so, by Lemma 8, $\mathfrak{A}$ is provable in HA.

THEOREM 7. *If a negative or prenex formula of* HA *is provable in Kleene's fragment of analysis, it is also provable in* HA *itself.*

PROOF. By Lemmas 7 and 9.

REMARK 13. It is well-known that second-order classical arithmetic *without* the comprehension axiom is an inessential extension of first-order classical arithmetic. The novelty of Theorem 7 lies essentially in the fact that the fan theorem schema of [9] does not yield new arithmetic theorems of the kind considered in Theorem 7.

REMARK 14. Note that the metamathematical methods used in the proof of Theorem 7 are finitist, and so we have a finitist consistency proof of Kleene's fragment of intuitionistic analysis relative to HA. — It is clear that, quite generally, the relative consistency of a formal system $S$ to $S'$ is proved finitistically, if $S$ is shown by finitist methods to be an inessential extension of $S'$ for some non-empty part of $S'$.[13] — The converse does not hold in general.

(i) Let $Z$ be classical first-order number theory, $P(n, m)$ its proof predicate, and let $\mathfrak{R}$ be Rosser's undecidable formula

$$(n)[P(n, \mathfrak{r}) \rightarrow (Em)_{<n}P(m, \mathfrak{r}_1)], \tag{17}$$

where $\mathfrak{r}$ is a suitable term whose value is the Gödel number of (17) and $\mathfrak{r}_1$ denotes a term whose value is the Gödel number of the negation of (17). Then, even in primitive recursive arithmetic, if Con $S$ is the arithmetization of '$S$ is consistent', $\mathrm{Con}(Z \cup \{\mathfrak{R}\})$ can be proved from Con $Z$. So, on the one hand $Z \cup \{\mathfrak{R}\}$ is proved to be consistent relatively to $Z$; on the other, it is not an inessential extension of $Z$ for universal formulae, since the universal formula $\mathfrak{R}$ itself is of course provable in $Z \cup \{\mathfrak{R}\}$, but not in $Z$.

(ii) Gödel's own argument showed that $\mathrm{Con}(Z \cup \{\neg \mathrm{Con}\, Z\})$ is provable in primitive recursive arithmetic from Con $Z$, but, since $\neg$Con $Z$ is a false existential formula, $Z \cup \{\neg \mathrm{Con}\, Z\}$ is not an inessential extension of $Z$ with respect to existential formulae. — It may be remarked that it is inessential for universal formulae. For, if $(n)A(n)$ is provable in $Z \cup \{\neg \mathrm{Con}\, Z\}$,

$$\vdash_Z \neg \mathrm{Con}\, Z \rightarrow (n)A(n). \tag{18}$$

But also, generally, for $A(n)$ primitive recursive, if $\vdash_S (n)A(n)$ then $\vdash_Z \mathrm{Con}\, S \rightarrow (n)A(n)$; and so $\vdash_Z \mathrm{Con}(Z \cup \{\neg \mathrm{Con}\, Z\}) \rightarrow (n)A(n)$, whence

$$\vdash_Z \mathrm{Con}\, Z \rightarrow (n)A(n). \tag{19}$$

[13] If the theorems of $S$ and $S'$ are recursively enumerable, and $S_1'$ is a non-empty recursively enumerable subspecies of $S'$, then we get a finitist relative consistency proof, provided we can prove in classical number theory that $S$ is an inessential extension of $S'$ with respect to $S_1'$. For, the latter statement is arithmetized by a formula $(n)(Em)R(n, m)$ with primitive recursive $R$, and such statements are provable by finitist methods if they are provable in classical arithmetic.

From (18) and (19), $\vdash_Z (n)A(n)$, as required.

Note that $\mathfrak{R}$ is weaker than Con $Z$ in the sense that $\vdash_Z \text{Con } Z \to \mathfrak{R}$ but not $\vdash_Z \mathfrak{R} \to \text{Con } Z$, while Gödel's undecidable formula is equivalent in $Z$ to Con $Z$.

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Note added February 22, 1961. (Cf. Remark 6.) To settle the relative importance of the two non-negative formulae above, note that the method of [12] Theorem 4 applies even if function symbols are added to HPC. Hence HPC *is weakly complete*, even in sence (1′), *for negations of prenex formulae whose quantifier-free part is negative.* Thus the conjunct $(x)[Q(x) \vee \neg Q(x)]$ is the main obstacle to proving the weak completeness of HPC for $\neg\mathfrak{A}$.