GROUP A. Postulates for the predicate calculus.

GROUP A1. Postulates for the propositional calculus.

1a. $A \supset (B \supset A)$.

1b. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$.

2. $\dfrac{A,\ A \supset B}{B}$.

3. $A \supset (B \supset A \,\&\, B)$.

4a. $A \,\&\, B \supset A$.

4b. $A \,\&\, B \supset B$.

5a. $A \supset A \vee B$.

5b. $B \supset A \vee B$.

6. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$.

7. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.

8°. $\neg\neg A \supset A$.

GROUP A2. (Additional) Postulates for the predicate calculus.

9. $\dfrac{C \supset A(x)}{C \supset \forall x A(x).}$

10. $\forall x A(x) \supset A(t)$.

11. $A(t) \supset \exists x A(x)$.

12. $\dfrac{A(x) \supset C}{\exists x A(x) \supset C.}$

GROUP B. (Additional) Postulates for number theory.

13. $A(0) \,\&\, \forall x(A(x) \supset A(x')) \supset A(x)$.

14. $a'=b' \supset a=b$.

15. $\neg a'=0$.

16. $a=b \supset (a=c \supset b=c)$.

17. $a=b \supset a'=b'$.

18. $a+0=a$.

19. $a+b'=(a+b)'$.

20. $a \cdot 0=0$.

21. $a \cdot b'=a \cdot b+a$.

(The reason for writing "°" on Postulate 8 will be given in § 23.)

One may verify that 14—21 are formulas; and that 1—13 (or in the case of 2, 9 and 12, the expression(s) above, and the expression below, the line) are formulas, for each choice of the A, B, C, or x, A(x), C, t, subject to the stipulations given at the head of the postulate list.

The class of 'axioms' is defined thus. A formula is an *axiom*, if it has one of the forms 1a, 1b, 3—8, 10, 11, 13 or if it is one of the formulas 14—21.

The relation of 'immediate consequence' is defined thus. A formula is an *immediate consequence* of one or two other formulas, if it has the form shown below the line, while the other(s) have the form(s) shown above the line, in 2, 9 or 12.

This is the basic metamathematical definition corresponding to Postulates 2, 9 and 12, but we shall restate it with additional terminology

which draws attention to the process of applying the definition. Postulates 2, 9 and 12 we call the *rules of inference*. For any (fixed) choice of the A and B, or the x, A(x) and C, subject to the stipulations, the formula(s) shown above the line is the *premise* (are the *first* and *second premise*, respectively), and the formula shown below the line is the *conclusion*, for the *application* of the rule (or the (*formal*) *inference* by the rule). The conclusion is an *immediate consequence* of the premise(s) (by the rule).

Carnap 1934 brings the two kinds of postulates under the common term 'transformation rules', by considering the axioms as the result of transformation from zero premises.

The definition of a '(formally) provable formula' or '(formal) theorem' can now be given inductively as follows.

1. If D is an axiom, then D is *provable*. 2. If E is *provable*, and D is an immediate consequence of E, then D is *provable*. 3. If E and F are *provable*, and D is an immediate consequence of E and F, then D is *provable*. 4. A formula is *provable* only as required by 1—3.

The notion can also be reached by using the intermediate concept of a '(formal) proof', thus. A (*formal*) *proof* is a finite sequence of one or more (occurrences of) formulas such that each formula of the sequence is either an axiom or an immediate consequence of preceding formulas of the sequence. A proof is said to be a proof *of* its last formula, and this formula is said to be (*formally*) *provable* or to be a (*formal*) *theorem*.

EXAMPLE 1. The following sequence of 17 formulas is a proof of the formula $a=a$. Formula 1 is Axiom 16. Formula 2 is an axiom, by an application of Axiom Schema 1a in which the A and the B of the schema are both $0=0$; and Formula 3 by an application in which the A is $a=b \supset (a=c \supset b=c)$ and the B is $0=0 \supset (0=0 \supset 0=0)$. Formula 4 is an immediate consequence of Formulas 1 and 3, as first and second premise respectively, by an application of Rule 2 in which the A of the rule is $a=b \supset (a=b \supset b=c)$ and the B is $[0=0 \supset (0=0 \supset 0=0)] \supset [a=b \supset (a=c \supset b=c)]$. Formula 5 is an immediate consequence of Formula 4, by an application of Rule 9 in which the x is c, the A(x) is $a=b \supset (a=c \supset b=c)$, and the C is $0=0 \supset (0=0 \supset 0=0)$ (which, note, does not contain the x free). Formula 9 is an axiom by an application of Axiom Schema 10, in which the x is a, the A(x) is $\forall b \forall c[a=b \supset (a=c \supset b=c)]$, and the t is $a+0$ (which, note, is free for the x in the A(x)). The A(t), by our substitution notation (§ 18), is the result of substituting the t for (the free occurrences of) the x in the A(x), i.e. here the A(t) is $\forall b \forall c[a+0=b \supset (a+0=c \supset b=c)]$.

$a=b \supset (a=c \supset b=c)$ — Axiom 16.

$0=0 \supset (0=0 \supset 0=0)$ — Axiom Schema 1a.

$\{a=b \supset (a=c \supset b=c)\} \supset \{[0=0 \supset (0=0 \supset 0=0)] \supset$
$[a=b \supset (a=c \supset b=c)]\}$ — Axiom Schema 1a.

$[0=0 \supset (0=0 \supset 0=0)] \supset [a=b \supset (a=c \supset b=c)]$ — Rule 2, 1, 3.

$[0=0 \supset (0=0 \supset 0=0)] \supset \forall c[a=b \supset (a=c \supset b=c)]$ — Rule 9, 4.

$[0=0 \supset (0=0 \supset 0=0)] \supset \forall b \forall c[a=b \supset (a=c \supset b=c)]$ — Rule 9, 5.

$[0=0 \supset (0=0 \supset 0=0)] \supset \forall a \forall b \forall c[a=b \supset (a=c \supset b=c)]$ — Rule 9, 6.

$\forall a \forall b \forall c[a=b \supset (a=c \supset b=c)]$ — Rule 2, 2, 7.

$\forall a \forall b \forall c[a=b \supset (a=c \supset b=c)] \supset \forall b \forall c\,[a+0=b \supset (a+0=c \supset b=c)]$
— Axiom Schema 10.

$\forall b \forall c[a+0=b \supset (a+0=c \supset b=c)]$ — Rule 2, 8, 9.

$\forall b \forall c[a+0=b \supset (a+0=c \supset b=c)] \supset \forall c[a+0=a \supset (a+0=c \supset a=c)]$
— Axiom Schema 10.

$\forall c[a+0=a \supset (a+0=c \supset a=c)]$ — Rule 2, 10, 11.

$\forall c[a+0=a \supset (a+0=c \supset a=c)] \supset [a+0=a \supset (a+0=a \supset a=a)]$ —
Axiom Schema 10.

$a+0=a \supset (a+0=a \supset a=a)$ — Rule 2, 12, 13.

$a+0=a$ — Axiom 18.

$a+0=a \supset a=a$ — Rule 2, 15, 14.

$a=a$ — Rule 2, 15, 16.

ꞔAMPLE 2. Let A be any formula. Then the following sequence of
formulas is a proof of the formula $A \supset A$. (In other words, what we
ɔit below is a 'proof schema', which becomes a particular proof on
:ituting any particular formula, such as $0=0$, for the metamathe-
ɔal letter "A"; and its last expression "$A \supset A$" is accordingly a
rem schema'.) Formula 1 is an axiom, by an application of Axiom
ma 1a in which the A and the B of the schema are the A of this
ıple. Formula 2 is an axiom, by an application of Axiom Schema 1b
ıich the A and the C of the schema are the A of this example, and
ȝ of the schema is the $A \supset A$ of this example. Formula 3 is an imme-
consequence of Formulas 1 and 2, as first and second premise,
ctively, by an application of Rule 2 in which the A of the rule
e $A \supset (A \supset A)$ of this example, and the B of the rule is the
$((A \supset A) \supset A)] \supset [A \supset A]$ of this example.

(1)

1. $A \supset (A \supset A)$ — Axiom Schema 1a.
2. $\{A \supset (A \supset A)\} \supset \{[A \supset ((A \supset A) \supset A)] \supset [A \supset A]\}$ — Axiom Schema 1b.
3. $[A \supset ((A \supset A) \supset A)] \supset [A \supset A]$ — Rule 2, 1, 2.
4. $A \supset ((A \supset A) \supset A)$ — Axiom Schema 1a.
5. $A \supset A$ — Rule 2, 4, 3.

The terms proof, theorem, etc. as defined for the formal system (i.e. formal proof, formal theorem, etc.) must be sharply distinguished from these terms in their ordinary informal senses, which we employ in presenting the metamathematics. A formal theorem is a formula (i.e. a certain kind of finite sequence of marks), and its formal proof is a certain kind of finite sequence of formulas. A metamathematical theorem is a meaningful statement about the formal objects, and its proof is an intuitive demonstration of the truth of that statement.

We mentioned three categories of formal objects (§ 16), but we shall be free to introduce others in the study of them, so long as the treatment is finitary. Besides this, a somewhat different extension of our subject matter occurs when we discuss the form of our metamathematical definitions and theorems in turn. If we chose to be meticulous in our way of doing this, it would constitute a metametamathematics. However, the same practice is common in (other branches of) informal mathematics; and we shall regard such discussions as incidental explanations, intended sometimes to make it easier to grasp quickly what is being done in the metamathematics, and sometimes to enable us to condense the statement of metamathematical theorems which could be stated without them.