MATHEMATICAL LOGIC

STEPHEN COLE KLEENE

Cyrus C. MacDuffee Professor of Mathematics
The University of Wisconsin, Madison

JOHN WILEY & SONS, INC.

New York · London · Sydney

CHAPTER I

THE PROPOSITIONAL CALCULUS

§ 1. Linguistic considerations: formulas. *Mathematical logic* (also called *symbolic logic*) is logic treated by mathematical methods. But our title has a double meaning, since we shall be studying the logic that is used in mathematics.

Logic has the important function of saying what follows from what. Every development of mathematics makes use of logic. A familiar example is the presentation of geometry in Euclid's "Elements" (c. 330–320 B.C.), in which theorems are deduced by logic from axioms (or postulates). But any orderly arrangement of the content of mathematics would exhibit logical connections. Similarly, logic is used in organizing scientific knowledge, and as a tool of reasoning and argumentation in daily life.

Now we are proposing to study logic, and indeed by mathematical methods. Here we are confronted by a bit of a paradox. For, how can we *treat* logic mathematically (or in any systematic way) without *using* logic in the treatment?

The solution of this paradox is simple, though it will take some time before we can appreciate fully how it works. We simply put the logic that we are studying into one compartment, and the logic that we are using to study it in another. Instead of "compartments", we can speak of "languages". When we are studying logic, the logic we are studying will pertain to one language, which we call the *object language*, because this language (including its logic) is an object of our study. Our study of this language and its logic, including our use of logic in carrying out the study, we regard as taking place in another language, which we call the *observer's language*.[1] Or we may speak of the *object logic* and the *observer's logic*.

It will be very important as we proceed to keep in mind this distinction between the logic we are studying (the object logic) and our use of logic in studying it (the observer's logic). To any student who is not ready to do so,

[1] In the literature, this language is usually called the "metalanguage" or the "syntax language". However, both these names often carry a connotation about the scope of the study or the type of the methods used in it. Cf. pp. 62–65 (especially bottom p. 63) of our "Introduction to Metamathematics" 1952b (hereafter cited as "IM").[10] To avoid such a connotation when it is not intended, we are adopting "observer's language".

In a textbook on Russian written in English, Russian is the object language and English is the observer's language.

we suggest that he close the book now, and pick some other subject instead, such as acrostics or beekeeping.

All of logic, like all of physics or all of history, constitutes a very rich and varied discipline. We follow the usual strategy for approaching such disciplines, by picking a small and manageable portion to treat first, after which we can extend our treatment to include some more.

The portion of logic we study first deals with connections between propositions which depend only on how some propositions are constructed out of other propositions that are employed intact, as building blocks, in the construction. This part of logic is called *propositional logic* or the *propositional calculus*.

We deal with propositions through declarative sentences which express them is some language (the object language); the propositions are the meanings of the sentences.[2] Declarative sentences express propositions (while interrogatory sentences ask questions and imperative sentences express commands). The same proposition may be expressed by different (declarative) sentences. Thus "John loves Jane" and "Jane is loved by John" express the same proposition, but "John loves Mary" expresses a different proposition. Under the usual definition of $>$ from $<$, the two sentences "$5 < 3$" and "$3 > 5$" express the same proposition (which happens to be false), namely that increasing 5 by a suitable positive quantity will give 3; but "$5^2 - 4^2 = 10$" expresses a different proposition (also false). Each of "$5 < 3$", "$3 > 5$" and "$5^2 - 4^2 = 10$" asserts something about the outcome of a mathematical process, which is the same process in the first two cases, but a different one in the third. "$3 - 2 = 1$" and "$(481 - 581) + 101 = 1$" express two different propositions (both true).

We save time, and retain flexibility for the applications, by not now describing any particular object language. (Examples will be given later.)

Throughout this chapter, we shall simply assume that we are dealing with one or another object language in which there is a class of (declarative) sentences, consisting of certain sentences (the aforementioned building blocks) and all the further sentences that can be built from them by certain operations, as we describe next. These sentences we call *formulas*, in deference to the use of mathematical symbolism in them or at least in our names for them.

First, in this language there are to be some unambiguously constituted sentences, whose internal structure we shall ignore (for our study of the propositional calculus) except for the purpose of identifying the sentences.

[2] Hence some writers call this part of logic "sentential logic" or the "sentential calculus".

We call *these* sentences *prime formulas* or *atoms*; and we denote them by capital Roman letters from late in the alphabet, as "P", "Q", "R", . . . , "P_1", "P_2", "P_3", ... Distinct such letters shall represent distinct atoms, each of which is to retain its identity throughout any particular investigation in the propositional calculus.

Second, the language is to provide five particular constructions or operations for building new sentences from given sentences. Starting with the prime formulas or atoms, we can use these operations, over and over again, to build other sentences, called *composite formulas* or *molecules*, as follows. (The prime formulas and the composite formulas together constitute the *formulas*.)[3] If each of A and B is a given *formula* (i.e. either a prime formula, or a composite formula already constructed), then A ∼ B, A ⊃ B, A & B and A ∨ B are (*composite*) *formulas*. If A is a given *formula*, then ¬A is a (*composite*) *formula*. (The first four operations are "binary" operations, the last is "unary".)

.The symbols "∼", "⊃", "&", "∨" and "¬" are called *propositional connectives*.[4] They can be read by using the words shown at the right in the following table; but the symbols are easier to write and manipulate.[5]

Equivalence	∼	"(is) equivalent (to)", "if and only if"
Implication	⊃	"implies", "if . . . then . . .", "only if"
Conjunction	&	"and"
Disjunction	∨	"or", ". . . or . . . or both", "and/or"
Negation	¬	"not"

Here we must mention the fact that the natural word languages, such as English, suffer from ambiguities. (Of this, more will be said later.) Logicians are therefore prone to build special symbolic languages. Our

[3] The analogy with chemistry limps a little, since we use "molecule" only for a formula which is not an atom, .whereas in chemistry an atom may sometimes be a molecule (e.g. helium He).

[4] Other symbols are often used in the literature, the most common being "≡", "↔" or "⇆" for our "∼"; "→" for "⊃"; "." (sometimes omitted) or "∧" for "&"; "∼", "–" or "¯" (thus: $\bar{A}$) for "¬". The symbols "∨" and "¬" need not be made as large as in the type used in this book.

[5] Anyone who doubts the advantages of symbols (in their proper place) is invited to solve the equation $x^2 + 3x - 2 = 0$ by completing the square (as taught in high school), *but* doing all the work in words. We start him off by stating the equation in words: The square of the unknown, increased by three times the unknown, and diminished by two, is equal to zero.

Anyone who doubts that *apt* choices of mathematical symbolism have played a major role in the modern development of mathematics and science is invited to multiply 416 by 144, *but* doing all the manipulations in Roman numerals. His problem is thus to multiply CDXVI by CXLIV. Cf. Time magazine, vol. 67 no. 12 (March 19, 1956), p. 83.

unspecified object language may be such a symbolic language, having symbols "$\sim$", "$\supset$", "&", "V" and "$\neg$" which play roles described accurately below but suggested or approximately described by the words. It comes to nearly the same thing to think of the object language as a suitably restricted and regulated *part* of a natural language, such as English; then "$\sim$", "$\supset$", "&", "V" and "$\neg$" can be thought of as names in the observer's language for the verbal expressions at the right in the table.[6]

The names at the left in the table above apply to the propositional connectives or to the formulas constructed using them. Thus "&" is our symbol for conjunction; and A & B is a conjunction, namely the conjunction of A and B. Also A $\supset$ B is the implication by A of B; etc.

So that there will be no ambiguity as to which formulas are the atoms, we now stipulate that none of the atoms be of any of the five forms A $\sim$ B, A $\supset$ B, A & B, A V B and $\neg$A which the molecules have.[7] For example, "Socrates is a man", "John loves Jane", "John loves Mary", "$5 < 3$", "$3 > 5$", "$a+b = c$" and "$a > 0$" (where "a", "b", "c" stand for numbers) could be atoms; then "John loves Jane or John loves Mary", "$\neg$ 5 < 3" and "5 < 3 $\sim$ 3 > 5" would be molecules.

We are using capital Roman letters from the beginning of the alphabet, as "A", "B", "C", . . . , "A_1", "A_2", "A_3", . . . , to stand for any formulas, not necessarily prime. Distinct such letters "A", "B", "C", . . . , "A_1",

[6] If "$\sim$", "$\supset$", etc. are symbols in the object language, then, when we write "A $\sim$ B", "A $\supset$ B", etc., we have a mixture of the two languages, since "A", "B" are names in the observer's language for formulas in the object language, while "$\sim$", "$\supset$", etc. are symbols of the object language itself. However, it should be clear enough here what is meant: "A $\supset$ B" is a name in the observer's language for the formula in the object language which results by infixing the symbol "$\supset$" of the object language between the two formulas in the object language which are named in the observer's language by "A" and "B" respectively. The mixing of languages disappears if we agree that "$\supset$" can serve as a name for itself in these contexts, and generally whenever we need a name in the observer's language for the symbol "$\supset$" of the object language. We say then that "$\supset$" is being used *autonymously* (after Carnap 1934).[10] We confine this usage to "$\sim$", "$\supset$", etc. and other *symbols* of a symbolic or partially symbolic object language when it should be clear that we are talking *about* expressions in that language.

Ordinary English words will not be used autonymously. Hence, when they are used outside of quotation marks, they must be understood as in the observer's language.

If we should wish to name the sentence A $\supset$ B but using the words "if . . . then . . ." instead of the symbol $\supset$ (here "$\supset$" is used autonymously), we would write " "if A then B" ". (The name of the sentence is what is inside the outer quotes; the whole is the name of that name.) Here again there is a mixture of languages (inside the inner quotes), but the meaning should be clear: the sentence named is the one obtained by replacing the letters "A" and "B" by the sentences they name.

[7] If any of the sentences we originally proposed to take as prime were already of those forms, we could start over by dissecting them into components not of those forms and using those components instead in our list of atoms.

"A_2", "A_3", . . . need *not* represent distinct formulas (in contrast to "P", "Q", "R", "P_1", "P_2", "P_3", . . . , which represent *distinct prime* formulas).

Composite formulas would sometimes be ambiguous as to the manner in which the symbols are to be associated if we did not introduce parentheses. So we shall write "$(A \supset B) \supset C$" or "$A \supset (B \supset C)$" and not simply "$A \supset B \supset C$". However we can minimize the need for parentheses by assigning decreasing ranks to our propositional connectives, in the order listed:[8]

$$\sim,\ \supset,\ \&,\ \vee,\ \neg.$$

Where there would otherwise be two ways of construing a formula, the connective with the greater rank reaches further. Thus "$A \supset B \;\&\; C$" shall mean $A \supset (B \;\&\; C)$, and "$C \sim A \;\&\; B \supset C$" shall mean $C \sim ((A \;\&\; B) \supset C)$. The unary operator $\neg$ being ranked last, "$\neg A \vee B$" shall mean $(\neg A) \vee B$ rather than $\neg(A \vee B)$, and "$\neg\neg A \supset A$" shall mean $(\neg(\neg A)) \supset A$. This practice is familiar in algebra, where "$a + bc^2 = d$" means $(a + (b(c^2))) = d$.

EXAMPLE 1. In "$A \supset (B \supset C)$" the letters "A", "B", "C" stand for formulas constructed from P, Q, R, . . . , P_1, P_2, P_3, . . . (i.e. from the atoms named by "P", "Q", "R", . . . , "P_1", "P_2", "P_3") using (zero or more times) $\sim$, $\supset$, &, $\vee$, $\neg$, and (as required) parentheses. For example, $A \supset (B \supset C)$ might be the particular formula $P \supset (Q \vee R \supset (R \supset \neg P))$ (so A is P, B is $Q \vee R$ and C is $R \supset \neg P$). Here a second pair of parentheses has been inserted, but our ranking of the symbols makes parentheses around $Q \vee R$ superfluous. The parentheses enable us to see how the formula was constructed, starting from the atoms P, Q, R, and using five steps of composition to introduce the five numbered *occurrences* of propositional connectives, thus:

$$
\begin{array}{lllll}
 & & & & P \\
 & Q \quad R & & R & \neg_2 P \\
 & Q \vee_1 R & & R \supset_3 & \neg_2 P \\
P & Q \vee_1 R \supset_4 & (R \supset_3 & \neg_2 P) & \\
P \supset_5 & (Q \vee_1 R \supset_4 & (R \supset_3 & \neg_2 P)) & \\
 & \text{—}1 \quad \text{—}1 & & \text{—}2 & \\
 & & \text{—}3 & \text{——}3 & \\
 & \text{————}4 & & \text{————}4 & \\
\text{—}5 & \text{————————}5 & & &
\end{array}
$$

[8] Some authors instead rank $\vee$ ahead of &; this is done in Algol and some other computer-programming languages. We shall rarely use our ranking between & and $\vee$ (which follows Hilbert and Bernays 1934 and IM).[1] Some authors (as Whitehead and Russell 1910–13) replace parentheses to a greater or lesser extent by dots ".", ":", ":." used in the manner of punctuation marks in English.

When we say (next) that, via the ranking, "$A \supset B \;\&\; C$" shall mean $A \;\&\; (B \supset C)$, we mean that "$A \supset B \;\&\; C$" becomes a name for the formula $A \;\&\; (B \supset C)$ which "$A \;\&\; (B \supset C)$" already named.

In a manner obvious from the parentheses or the construction, each occurrence of a connective "connects" or "applies to" or "operates on" one or two parts of the formula, called the *scope* of that (occurrence of a) connective. The scopes of the connectives are shown here by correspondingly numbered underlines; thus the scope of $\supset_4$ consists of the two parts $Q \vee R$ and $R \supset \neg P$.

EXERCISE 1.1. Identify the scope of each (occurrence of a) propositional connective: (a) $P \supset \neg P \sim \neg P$. (b) $\neg P \,\&\, Q \sim R \,\&\, \neg\neg(P \vee Q) \supset S$.

§ 2. Model theory: truth tables, validity. Not only are we restricting ourselves in this chapter to the study of the logic of propositions. But also in this and later chapters we shall concern ourselves primarily with a certain kind of logic, called *classical logic*.

Since the discovery of non-Euclidean geometries by Lobatchevsky (1829) and Bolyai (1833), it has been clear that different systems of geometry are conceptually equally possible. (We shall say a little more about this in § 36.) Similarly, there are different systems of logic. Different theories can be deduced from the same mathematical postulates, the differences depending on the system of logic used to make the deductions. The classical logic, like the Euclidean geometry, is the simplest and the most commonly used in mathematics, science and daily life. In this book we shall find the space for only brief indications of other kinds of logic.[9]

Thus far we have assumed about each prime formula or atom only that it can be identified; i.e. that each time it occurs it can be recognized as the same, and as different from other atoms.

Now we make one further assumption about the atoms, which is characteristic of classical logic. We assume that each atom (or the proposition it expresses) is either *true* or *false* but not both.

We are not assuming that we *know* of each atom *whether* it is true or false. That knowledge would require us to look into the constitution of the atoms, or to consider facts to which they allude under an agreed interpretation of the words or symbols, none of which is within our purview *in the propositional calculus*.

Our assumption is thus that, for each atom, there are exactly two possibilities: it may be true, it may be false.

The question now arises: How does the truth or falsity (*truth value*) of a composite formula or molecule depend upon the truth value(s) of its component prime formula(s) or atom(s)? This will be determined by repeated use of five definitions, given by the following tables. These tables relate the truth value of each molecule to the truth value(s) of its *immediate*

[9] *Some* other kinds of logic would require other propositional connectives than the five we introduced in § 1, e.g. □ and ◊ end § 12.

component(s). In the left-hand columns we list all the possible *assignments* of truth t and falsity f to the immediate component(s). Then in the line (or row) for a given assignment, we show the resulting truth value of each molecule in the column headed by that molecule.

A	B	$A \sim B$	$A \supset B$	$A \& B$	$A \vee B$
t	t	t	t	t	t
t	f	f	f	f	t
f	t	f	t	f	t
f	f	t	t	f	f

A	$\neg A$
t	f
f	t

Thus $A \sim B$ is true exactly when A and B have the same truth value (hence the reading "equivalent", i.e. "equal valued", for $\sim$); $A \supset B$ is false exactly when A is true and B is false; $A \& B$ is true exactly when A and B are both true; $A \vee B$ is false exactly when both are false; and $\neg A$ is true exactly when A is false.

Some controversy has arisen about the name "implication", and the reading "implies", for our $\supset$. Say A is "the moon is made of green cheese" and B is "$2+2=5$". Then $A \supset B$ is true under our table (because A is false), even though there is no connection of ideas between A and B. Similarly, if B is "$2+2=4$", $A \supset B$ is true (because B is true), quite apart from whether A bears any relationship to "$2+2=4$". This is considered paradoxical by some writers (Lewis 1912, 1917, Lewis and Langford 1932).[10]

In modern mathematics the name "multiplication" is often used for various mathematical operations that behave more or less analogously to the arithmetical one called "multiplication". Similarly, we find it convenient to use the name "implication" to designate the operation defined by the second truth table above; and then in our logical discussions we usually read $A \supset B$ as "A implies B", even though "if A then B" or "A only if B" probably renders the meaning better in everyday English. This "implication", and the present "equivalence", are called more specifically "material implication" and "material equivalence".[11]

It is, of course, possible to be interested in other senses of "implication"; but then one must have recourse to ways of defining it other than by a

[10] A date appearing in conjunction with a person's name ordinarily constitutes a reference to the bibliography at the end of the book. The few exceptions are dates of old and well-known works not primarily in logic and dates not associated with publication.

[11] In everyday English, ". . . if . . . then" functions grammatically as a conjunction like "and" and "or" (connecting sentences), while "implies" is a transitive verb (connecting nouns). From this standpoint, when we use "if A then B" and "A implies B" interchangeably, we can regard the latter as short for " "A" implies "B" " or "that A implies that B".

"two-valued truth table". Our definition is the only reasonable one with such a table.[12]

A related question is why we should want to assert a material implication A ⊃ B, when if A is true we could more simply assert B (or if our hearers don't know that A is true, we could more informatively assert A & B), and if A is false we could more simply say nothing. But we ordinarily assert sentences of the form "If A, then B" when we don't know whether A is true or not. For example, before an election I might say [1] "If our candidate for President carries the state by 500,000, then our man for the Senate will also win". This form of statement enables me to predict what will happen in *one* eventuality without attempting to say more. If it turns out that our candidate for President doesn't carry the state by 500,000, my prediction will not have been proved false. Since we are committed here to a two-valued logic, my statement should then be regarded as true, though perhaps uninteresting. If when I speak, returns are in showing our candidate ahead by a safe 500,000, I would more likely say [1a] "Our man for the Senate will win" or [1b] "Since our candidate for President is carrying the state by 500,000, our man for the Senate will win". But [1] would not have become false, just partially redundant and thus unnatural for me to say (unless I haven't heard the latest election returns).

To take an analogous mathematical example, suppose a positive integer $n > 1$ is written on a piece of paper in *your* pocket, and I do not know what the integer is. I could truthfully say [2] "If n is odd, then $x^n + y^n$ can be factored". By saying this, I am claiming that, *when* you produce the paper showing the value of n, then I will be able to factor $x^n + y^n$ for the n you produce if it turns out to be odd (and I am making *no claim* about the factorability of $x^n + y^n$ in the contrary case). Thus, if you have bet that I am wrong, to settle the bet you produce the number n. If for example it is 3, I then show you the factorization $(x + y)(x^2 - xy + y^2)$, and you pay me. If for example it is 4 (or 6), I win automatically.

These examples should make it clear that material implication A ⊃ B ("If A, then B") is a useful and natural form of expression.[13] Similar remarks apply to material equivalence A ∼ B.

[12] Then ordinary usage certainly requires "If A, then B" to be true when A and B are both true, and to be false when A is true but B is false. So only our choice of t in the third and fourth lines can be questioned. But if we changed t to f in both these lines, we would simply get a synonym for &; in the third line only, for ∼. If we changed t to f in the fourth line only, we would lose the useful property of our implication that "If A, then B" and "If not B, then not A" are true under exactly the same circumstances (which will appear later as *12a in Theorem 2).

[13] We are talking about the use of "If . . . , then . . ." with verbs in the indicative mood. Grammar allows also contrary-to-fact conditionals with verbs in the subjunctive mood. These are "If A, then B" sentences where the falsity of A (which is indicated) does not make the whole true irrespective of what B is. Say the $n > 1$ in your pocket

Likewise, when we don't know whether A is true or not, and don't know whether B is true or not, it can be useful to assert "A or B" or in symbols A ∨ B. If we already knew that A is true, it would be simpler and more informative to say "A"; etc. Our disjunction A ∨ B, defined by the fourth truth table, is the "inclusive disjunction" or "nonexclusive disjunction", which is true when A is true or B is true or both A and B are true. This is more useful to us than the "exclusive disjunction", expressed by the words "A or B but not both", which instead has f also in the first row. While English is ambiguous, Latin is clear, using "vel" for the inclusive disjunction and "aut" for the exclusive disjunction. The symbol ∨ comes from the first letter of "vel".

We postpone further discussion of the relation between our symbols and ordinary language to the end of the chapter.

We now illustrate the repeated use of the above tables by computing the truth table for P ⊃ (Q ∨ R ⊃ (R ⊃ ¬P)). The result is shown first as (1) below, then the details of the computation of the third line (or row). For this line, we first substitute for the atoms P, Q, R the respective values t, f, t assigned to them for that line. Then we compute the values of innermost composite parts repeatedly. Thus by the table for ∨, f ∨ t is t; by the table for ¬, ¬t is f; and by the table for ⊃, t ⊃ f is f (which we use three times). The successive stages in this computation are shown in successive lines for clarity, and are then summarized in a single line.

Completed truth table:

(1)

	P	Q	R	P ⊃ (Q ∨ R ⊃ (R ⊃ ¬P))
1.	t	t	t	f
2.	t	t	f	t
3.	t	f	t	f
4.	t	f	f	t
5.	f	t	t	t
6.	f	t	f	t
7.	f	f	t	t
8.	f	f	f	t

is already known to me and is 4. I could say truthfully [2′] "If the n had been odd, I could have factored $x^n + y^n$"; but I could not say truthfully [2″] "If the n had been odd, I could have factored $x^{n+1} + y^{n+1}$". (With n odd, $x^{n+1} + y^{n+1}$ may or may not be factorable.) A contrary-to-fact conditional "If A, then B" makes an assertion about a "hypothetical situation" analogous to the actual situation but differing from it by A holding. Sunday morning quarterbacks and Wednesday morning politicians find them useful. We gave a mathematical example, because we can be positive in affirming [2′] and refraining from affirming [2″], whereas in football and politics matters are more controversial.

Computation of the third line:

$$
\begin{array}{l}
P \supset (Q \vee R \supset (R \supset \neg P)) \\
t \supset (f \vee t \supset (t \supset \neg t)) \\
t \supset (\quad t \quad \supset (t \supset f\ )) \\
t \supset (\quad t \quad \supset \quad f \quad) \\
t \supset \qquad\quad f \\
\quad f \\
\hline
t\ f\ \ f\ t\ t\ f\ \ t\ f\ \ f\ t
\end{array}
$$

The computation process illustrated just now constitutes a mechanical procedure by which we can compute the truth table for any formula E, or more specifically the truth table for E using (or "entered from") a given list $P_1, \ldots, P_n$ of the prime components of E. (In (1), we used the list P, Q, R; we could have used instead Q, P, R or Q, R, P, etc. to obtain different tabulations of the same collection of eight computation results.) In the trivial case that E is a prime formula P, the computation takes zero steps, and the value column is identical with the column of assignments to P.

In practice it is not always necessary to apply the procedure in full detail. Thus the observation that $A \supset B$ is t whenever A is f (irrespective of the truth value of B) suffices to justify our entering t in the last four lines of the above table without further ado.

There are formulas for which the value column in the truth table will contain only t's, for example $P \,\&\, \neg P \supset (Q \vee R \supset (R \supset \neg P))$, $P \supset \neg P \sim \neg P$ and $P \supset P$, as the reader may verify (Exercise 2.2). The order of listing the prime components does not matter here. (Why?) Such formulas are therefore always true, regardless of the truth or falsity of their prime components. Without knowing the truth values of the prime components, we can nevertheless say that the composite formula is true. Such formulas are said to be *valid*, or to be *identically true*, or (after Wittgenstein 1921) to be *tautologies* (in, or of, the propositional calculus).

To give a verbal example, the proposition "If I am going too fast, then I am going too fast" is true on the basis of the propositional calculus; indeed, it has the form $P \supset P$. But the proposition "I am going too fast", if true, is true on other grounds.

It might seem that the valid formulas or tautologies are the least interesting, because from one point of view they give no information. My admitting that "If I am going too fast, then I am going too fast" can hardly give any of you much satisfaction. But it will appear as we proceed that the tautologies are important.

EXERCISES. 2.1. Find the truth tables: (a) $\neg P \vee Q$. (Compare this with the table for $P \supset Q$.) (b) $(\neg P \vee Q) \,\&\, (R \supset (P \sim Q))$. (c) $Q \supset P \vee Q$. Are any of these formulas valid?

2.2. Verify that $P \,\&\, \neg P \supset (Q \vee R \supset (R \supset \neg P))$, $P \supset \neg P \sim \neg P$ and $P \supset P$ are valid.

2.3. Show that the following formulas are valid. To reduce the work, observe that the whole implication $A \supset B$ fails to be valid only if you can pick truth values for P, Q (or for P, Q, R, S) which simultaneously make B take the value $\mathfrak{f}$ and A the value $\mathfrak{t}$. Consider all the choices of values that make B $\mathfrak{f}$, and verify that none of them make A $\mathfrak{t}$.

(a) $((P \supset Q) \supset P) \supset P$. (Peirce's law, 1885.)

(b) $((P \supset R) \,\&\, (Q \supset S)) \,\&\, (\neg R \vee \neg S) \supset \neg P \vee \neg Q$.

(c) $(P \supset Q) \supset (\neg Q \supset \neg P)$.[14]

2.4. Show the following not valid by computing just one suitable line of the table: (a) $P \vee Q \supset P \,\&\, Q$. (b) $(P \supset Q) \supset (Q \supset P)$.[14]

2.5. Find formulas composed from P, Q, R whose truth tables have the following value columns: (a) $\mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{t}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}$. (Use a method applicable to any truth table with just one $\mathfrak{t}$.) (b) $\mathfrak{t}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{t}\ \mathfrak{f}\ \mathfrak{t}\ \mathfrak{f}$. (First use a method applicable to any table with more than one $\mathfrak{t}$. Can you find a shorter formula with the same table?) (c) $\mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}\ \mathfrak{f}$.

§ 3. Model theory: the substitution rule, a collection of valid formulas. The definition of validity provides us with an automatic way of deciding as to the validity of any formula: simply compute its truth table, and see whether we get all $\mathfrak{t}$'s. This is a very fortunate situation, and one should not hesitate to do this in any case of doubt.

However, computing truth tables of formulas at random would be a rather slow way of discovering valid formulas. Anyone not familiar with simple examples of valid formulas and with methods for proceeding to others (whether or not he has officially studied logic) would properly be described as sluggish in his mental processes.

One simple principle is this. In *defining* validity, we use a truth table entered from the prime components, so as to take into account all the structure of the formula available to the propositional calculus. However, to *establish* validity, we may not need to dissect a formula all the way down to its prime components or atoms. If we get all $\mathfrak{t}$'s in a table entered from (values of) components not necessarily prime, we can be sure it is valid. For example, $P \,\&\, \neg P \supset P \,\&\, \neg P$ is of the form $A \supset A$; Table (a) (below) entered from A gives all $\mathfrak{t}$'s; hence the formula is valid. For, in computing each line of Table (b) entered from P (as is called for under the definition of validity), the first part of the computation consists in

[14] $\neg Q \supset \neg P$ is the *contrapositive* of $P \supset Q$, and $Q \supset P$ is the *converse* of $P \supset Q$.

computing the value of P & ¬P, i.e. of the A. Then the rest of the computation consists in computing the value of the whole from that of A (shown underlined in Table (b)); but this we have already done with result t in computing Line 2 of Table (a). Table (a) is the same as Table (c)

(a)

A	A	⊃	A
t	t	t	t
f	$\underline{f}$	$\underline{t}$	$\underline{f}$

(b)

P	P	&	¬	P	⊃	P	&	¬	P
t	t	$\underline{f}$	f	t	$\underline{t}$	t	$\underline{f}$	f	t
f	f	$\underline{f}$	t	f	$\underline{t}$	f	$\underline{f}$	t	f

(c)

P	P	⊃	P
t	t	t	t
f	f	t	f

except for the notation; instead of saying we construct a table for A ⊃ A entered from A, it comes to the same thing to say that we verify the validity of P ⊃ P, and then substitute A (i.e. P & ¬P) for P in P ⊃ P. This reasoning gives the following theorem, in which we write "⊨ E" as a short way of saying "E is valid".[15]

Theorem 1. (Substitution for atoms.) *Let* E *be a formula containing only the atoms* $P_1, \dots, P_n$, *and let* E* *come from* E *by substituting formulas* $A_1, \dots, A_n$ *simultaneously for* $P_1, \dots, P_n$, *respectively. If* ⊨ E, *then* ⊨ E*.

On the other hand, to show by truth tables that a formula is *not* valid, the tables must in general be entered from the prime components. For example, P & ¬P ⊃ Q is of the form A ⊃ B. The table for A ⊃ B entered from A and B (§ 2) does not have all t's (in other words, P ⊃ Q is not valid). But P & ¬P ⊃ Q is valid. This example shows that the converse of Theorem 1, namely "If ⊨ E*, then ⊨ E", does not hold.

Returning to the example preceding Theorem 1, since Table (a) entered from A has all t's (or equivalently (c) has all t's), we shall have all t's for *every* formula of the form A ⊃ A, not just the particular one we took there with P & ¬P as the A. This is included in the theorem; for, with E fixed and "⊨ E" established, we can apply the theorem with any choice of $A_1, \dots, A_n$.

[15] Expressions containing "⊨" ("⊨ E" here and "$A_1, \dots, A_m$ ⊨ B" in § 7) are not formulas of the object language, but expressions of the observer's language, used in writing concisely certain statements about formulas. The definition of "formula" for the propositional calculus was concluded in § 1, and allows only ∼, ⊃, &, V, ¬ (as symbols of the object language) to be used in building up formulas from the atoms P, Q, R, . . . , $P_1, P_2, P_3, \dots$. Now "⊨" is a symbol of the observer's language, and hence stands outside every formula, and outranks ∼, ⊃, &, V, ¬; thus "⊨ A ∼ B" means "⊨ (A ∼ B)" rather than "(⊨ A) ∼ B".

We use this principle to establish in the next theorem a collection of forms of valid formulas.[16]

For example, as *1 we give the result just established.

As 5b, we claim that, for each choice of formulas (built up from P, Q, R, . . . , P_1, P_2, P_3, . . .) as the A and B, the resulting formula B ⊃ A ∨ B is valid. For, in Exercise 2.1 (c) we saw that Q ⊃ P ∨ Q is valid; and hence by Theorem 1 B ⊃ A ∨ B is valid.

In the same way, every one of the results in Theorem 2 can be proved automatically, by first verifying the validity of the particular formula which has P, Q, R in place of A, B, C, and then using Theorem 1 (or equivalently, by constructing a truth table entered from A, B, C).

The student may accordingly take the whole list on faith, as he does a table of square roots or trigonometric functions or integrals.

We intend that the student should acquire the ability to use these results, and indeed should learn enough of them so that he can operate without Theorem 2 before him. However, we do not ask him to learn the list outright *now*, but rather to use it for reference and in doing so to become familiar with the results most often used.[17]

THEOREM 2. *For any choice of formulas* A, B, C:

1a. ⊨ A ⊃ (B ⊃ A).
1b. ⊨ (A ⊃ B) ⊃ ((A ⊃ (B ⊃ C)) ⊃ (A ⊃ C)).
3. ⊨ A ⊃ (B ⊃ A & B).
4a. ⊨ A & B ⊃ A.
4b. ⊨ A & B ⊃ B.
5a. ⊨ A ⊃ A ∨ B.
5b. ⊨ B ⊃ A ∨ B.
6 . ⊨ (A ⊃ C) ⊃ ((B ⊃ C) ⊃ (A ∨ B ⊃ C)).

[16] Most of these results have equivalents in IM (= Kleene "Introduction to Metamathematics" 1952b). With a few exceptions, we employ the numbering of IM. This will facilitate using IM as a reference work supplementing the present book, or the present book as an introduction to IM. (This accounts for the gaps and other irregularities in the numbering in Theorem 2. The present 9a, 10a, 10b, *4a, *12a, *55c, *63a do not correspond to like-numbered results in IM; and *55a, *55b correspond to *63, *62 of IM, but are renumbered here to come earlier.)

The meaning of "°" on 8, *12a, etc. will be explained at the end of § 12.

[17] The developments below should assist the reader in becoming familiar with these and other results. We shall make various applications, and establish interconnections, which should help to fix them in mind. For some of them we shall give new proofs to make the results more meaningful.

Following Church 1956 p. 73, *49 may be called more specifically the "complete law of double negation"; 8 the "law of double negation" simply; and the converse of 8 the "converse law of double negation".[14] Similarly, *12a is the "complete law of contraposition"; with ~ replaced by ⊃, the "law of contraposition"; by ⊂, the "converse law of contraposition". Also, 1a is the "law of affirmation of the consequent" (cf. *10a); and *7 is the "law of reductio ad absurum". With ~ replaced by ⊃, *4a is "importation"; by ⊂, "exportation". (When replacing ~ here, supply parentheses.)

7. $\vDash (A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.

8°. $\vDash \neg\neg A \supset A$.

9a. $\vDash (A \supset B) \supset ((B \supset A) \supset (A \sim B))$.

10a. $\vDash (A \sim B) \supset (A \supset B)$.

10b. $\vDash (A \sim B) \supset (B \supset A)$.

(Introductions and eliminations of logical symbols.)

*1. $\vDash A \supset A$.

*2. $\vDash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$.

*3. $\vDash A \supset (B \supset C) \sim B \supset (A \supset C)$.

*4a. $\vDash A \supset (B \supset C) \sim A \,\&\, B \supset C$.

(Principle of identity, chain inference, interchange of premises, importation and exportation.)

*10a. $\vDash \neg A \supset (A \supset B)$.

*12a°. $\vDash A \supset B \sim \neg B \supset \neg A$.

(Denial of the antecedent, contraposition.)

*19. $\vDash A \sim A$.

*20. $\vDash (A \sim B) \sim (B \sim A)$.

*21. $\vDash (A \sim B) \,\&\, (B \sim C) \supset (A \sim C)$.

(Reflexive, symmetric and transitive properties of equivalence.)

*31. $\vDash (A \,\&\, B) \,\&\, C \sim A \,\&\, (B \,\&\, C)$.

*32. $\vDash (A \vee B) \vee C \sim A \vee (B \vee C)$.

*33. $\vDash A \,\&\, B \sim B \,\&\, A$.

*34. $\vDash A \vee B \sim B \vee A$.

*35. $\vDash A \,\&\, (B \vee C) \sim (A \,\&\, B) \vee (A \,\&\, C)$.

*36. $\vDash A \vee (B \,\&\, C) \sim (A \vee B) \,\&\, (A \vee C)$.

*37. $\vDash A \,\&\, A \sim A$.

*38. $\vDash A \vee A \sim A$.

*39. $\vDash A \,\&\, (A \vee B) \sim A$.

*40. $\vDash A \vee (A \,\&\, B) \sim A$.

(Associative, commutative, distributive, idempotent and elimination laws.)

*49°. $\vDash \neg\neg A \sim A$.

*50. $\vDash \neg(A \,\&\, \neg A)$.

*51°. $\vDash A \vee \neg A$.

(Law of double negation, denial of contradiction, law of the excluded middle.)

*55a. $\vDash \neg(A \vee B) \sim \neg A \,\&\, \neg B$.

*55b°. $\vDash \neg(A \,\&\, B) \sim \neg A \vee \neg B$.

*55c°. $\vDash \neg(A \supset B) \sim A \,\&\, \neg B$.

(De Morgan's laws 1847,[18] negation of an implication.)

*56°. $\vDash A \vee B \sim \neg(\neg A \,\&\, \neg B)$.

*57°. $\vDash A \,\&\, B \sim \neg(\neg A \vee \neg B)$.

*58°. $\vDash A \supset B \sim \neg(A \,\&\, \neg B)$.

*59°. $\vDash A \supset B \sim \neg A \vee B$.

*60°. $\vDash A \,\&\, B \sim \neg(A \supset \neg B)$.

*61°. $\vDash A \vee B \sim \neg A \supset B$.

*63a. $\vDash (A \sim B) \sim (A \supset B) \,\&\, (B \supset A)$.

(Expressions for some connectives in terms of others.)

[18] In verbal form, these go back at least to Ockham ("Summa Logicae", 1323–9). Cf. Łukasiewicz 1934, Bocheński 1956.