Begriffsschrift, *a formula language, modeled upon that of arithmetic, for pure thought*

GOTTLOB FREGE

(*1879*)

This is the first work that Frege wrote in the field of logic, and, although a mere booklet of eighty-eight pages, it is perhaps the most important single work ever written in logic. Its fundamental contributions, among lesser points, are the truth-functional propositional calculus, the analysis of the proposition into function and argument(s) instead of subject and predicate, the theory of quantification, a system of logic in which derivations are carried out exclusively according to the form of the expressions, and a logical definition of the notion of mathematical sequence. Any single one of these achievements would suffice to secure the book a permanent place in the logician's library.

Frege was a mathematician by training;[a] the point of departure of his investigations in logic was a mathematical question, and mathematics left its mark upon his logical accomplishments. In studying the concept of number, Frege was confronted with difficulties when he attempted to give a logical analysis of the notion of sequence. The imprecision and ambiguity of ordinary language led him to look for a more appropriate tool; he devised a new mode of expression, a language that deals with the "conceptual content" and that he came to call "Begriffsschrift".[b] This ideography is a "formula language", that is, a *lingua characterica*, a language written with special symbols, "for pure thought", that is, free from rhetorical embellishments, "modeled upon that of arithmetic", that is, constructed from specific symbols that are manipulated according to definite rules. The last phrase does not mean that logic mimics arithmetic, and the analogies, uncovered by Boole and others, between logic and arithmetic are useless for Frege, precisely because he wants to employ logic in

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[a] See his *Inaugural-Dissertation* (*1873*) and his thesis for *venia docendi* (*1874*).

[b] In the translation below this term is rendered by "ideography", a word used by Jourdain in a paper (*1912*) read and annotated by Frege; that Frege acquiesced in its use was the reason why ultimately it was adopted here. Another acceptable rendition is "concept writing", used by Austin (*Frege 1950*, p. 92e).

Professor Günther Patzig was so kind as to report in a private communication that a student of his, Miss Carmen Diaz, found an occurrence of the word "Begriffsschrift" in Trendelenburg (*1867*, p. 4, line 1), a work that Frege quotes in his preface to *Begriffsschrift* (see below, p. 6). Frege used the word in other writings, and in particular in his major work (*1893*, *1903*), but subsequently he seems to have become dissatisfied with it. In an unpublished fragment dated 26 July 1919 he writes: "I do not start from concepts in order to build up thoughts or propositions out of them; rather, I obtain the components of a thought by decomposition [[Zerfällung]] of the thought. In this respect my Begriffsschrift differs from the similar creations of Leibniz and his successors—in spite of its name, which perhaps I did not chose very aptly".

order to provide a foundation for arithmetic. He carefully keeps the logical symbols distinct from the arithmetic ones. Schröder (*1880*) criticized him for doing just that and thus wrecking a tradition established in the previous thirty years. Frege (*1882*, pp. 1–2) answered that his purpose had been quite different from that of Boole: "My intention was not to represent an abstract logic in formulas, but to express a content through written signs in a more precise and clear way than it is possible to do through words. In fact, what I wanted to create was not a mere *calculus ratiocinator* but a *lingua characterica* in Leibniz's sense".

Mathematics led Frege to an innovation that was to have a profound influence upon modern logic. He observes that we would do violence to mathematical statements if we were to impose upon them the distinction between subject and predicate. After a short but pertinent critique of that distinction, he replaces it by another, borrowed from mathematics but adapted to the needs of logic, that of function and argument. Frege begins his analysis by considering an ordinary sentence and remarks that the expression remains meaningful when certain words are replaced by others. A word for which we can make such successive substitutions occupies an argument place, and the stable component of the sentence is the function. This, of course, is not a definition, because in his system Frege deals not with ordinary sentences but with formulas; it is merely an explanation, after which he introduces functional letters and gives instructions for handling them and their arguments. Nowhere in the present text does Frege state what a function is or speak of the value of a function. He simply says that a judgment is obtained when the argument places between the parentheses attached to a functional letter have been properly filled (and, should the case so require, quantifiers have been properly used). It is only in his subsequent writings (*1891* and thereafter) that Frege will devote a great deal of attention to the nature of a function.

Frege's booklet presents the propositional calculus in a version that uses the conditional and negation as primitive connectives. Other connectives are examined for a moment, and their intertranslatability with the conditional and negation is shown. Mostly to preserve the simple formulation of the rule of detachment, Frege decides to use these last two. The notation that he introduces for the conditional has often been criticized, and it has not survived. It presents difficulties in printing and takes up a large amount of space. But, as Frege himself (*1896*, p. 364) says, "the comfort of the typesetter is certainly not the *summum bonum*", and the notation undoubtedly allows one to perceive the structure of a formula at a glance and to perform substitutions with ease. Frege's definition of the conditional is purely truth-functional, and it leads him to the rule of detachment, stated in § 6. He notes the discrepancy between this truth-functional definition and ordinary uses of the word "if". Frege dismisses modal considerations from his logic with the remark that they concern the grounds for accepting a judgment, not the content of the judgment itself. Frege's use of the words "affirmed" and "denied", with his listing of all possible cases in the assignment of these terms to propositions, in fact amounts to the use of the truth-table method. His axioms for the propositional calculus (they are not independent) are formulas (1), (2), (8), (28), (31), and (41). His rules of inference are the rule of detachment and an unstated rule of substitution. A number of theorems of the propositional calculus are proved, but no question of completeness, consistency, or independence is raised.

Quantification theory is introduced in § 11. Frege's instructions how to use italic and German letters contain, in effect, the rule of generalization and the rule that allows us to infer $A \supset (x)F(x)$ from $A \supset F(x)$ when x does not occur free in A. There are three new axioms: (58) for instantiation, (52) and (54) for identity. No rule of substitution is explicitly stated, and one has to examine Frege's practice in his derivations to see what he allows. The substitutions are indicated by tables on the left of the derivations. These substitutions are simultaneous substitutions. When a substitution is specified with the help of "Γ", which plays the role of what we would today call a syntactic variable, particular care should be exercised, and it proves convenient to perform the substitutions that do not involve "Γ" before that involving "Γ" is carried out. The point will become clear to the reader if he compares, for example, the derivation of (51) with that of (98). Frege's derivations are quite detailed and, even in the absence of an explicit rule of substitution, can be unambiguously reconstructed.

Frege allows a functional letter to occur in a quantifier (p. 24 below). This license is not a necessary feature of quantification theory, but Frege has to admit it in his system for the definitions and derivations of the third part of the book. The result is that the difference between function and argument is blurred. In fact, even before coming to quantification over functions, Frege states (p. 24 below) that we can consider $\Phi(A)$ to be a function of the argument Φ as well as of the argument A. (This is precisely the point that Russell will seize upon to make it bear the brunt of his paradox—see below, p. 125). It is true that Frege writes (p. 24 below) that, if a functional letter occurs in a quantifier, "this circumstance must be taken into account". But the phrase remains vague. The most generous interpretation would be that, in the scope of the quantifier in which it occurs, a functional letter has to be treated as such, that is, must be provided with a pair of parentheses and one or more arguments. Frege, however, does not say as much, and in the derivation of formula (77) he substitutes $\mathfrak{F}$ for $\mathfrak{a}$ in $f(\mathfrak{a})$, at least as an intermediate step. If we also observe that in the derivation of formula (91) he substitutes $\mathfrak{F}$ for f, we see that he is on the brink of a paradox. He will fall into the abyss when (*1891*) he introduces the course-of-values of a function as something "complete in itself", which "may be taken as an argument". For the continuation of the story see pages 124–128.

This flaw in Frege's system should not make us lose sight of the greatness of his achievement. The analysis of the proposition into function and argument, rather than subject and predicate, and quantification theory, which became possible only after such an analysis, are the very foundations of modern logic. The problems connected with quantification over functions could be approached only after a quantification theory had already been established. When the slowness and the wavering that marked the development of the propositional calculus are remembered, one cannot but marvel at seeing quantification theory suddenly coming full-grown into the world. Many years later (*1894*, p. 21) Peano still finds quantification theory "abstruse" and prefers to deal with it by means of just a few examples. Frege can proudly answer (*1896*, p. 376) that in 1879 he had already given all the laws of quantification theory; "these laws are few in number, and I do not know why they should be said to be abstruse".

In distinguishing his work from that of his predecessors and contemporaries, Frege repeatedly opposes a *lingua characterica* to a *calculus ratiocinator*. He uses these terms, suggested by Leibniz, to bring out an important feature of his system, in fact, one of the greatest achievements of his *Begriffsschrift*. In the pre-Fregean calculus of propositions and classes, logic, translated into formulas,

(2) On page 29 of the German text, in § 15, the letters to the left of the long vertical line under (1) should be "a" and "b", not "a" and "b";

(3) The misprint indicated in footnote 18, p. 57 below;

(4) The misprint indicated in footnote 21, p. 65 below.

Moreover, Misprint 3 in the reprint's list does not occur in the German text used for the present translation; apparently, it is not a misprint at all but is simply due to the poor printing of some copies. The reprint also introduces misprints of its own: on page 1, line 4u, we find "——" where there should be "$\vdash$——"; on page 62, near the top of the page, "$\genfrac{}{}{0pt}{}{\gamma}{\beta}$" should be "$\genfrac{}{}{0pt}{}{\gamma}{\overset{\smile}{\beta}}$".; on page 65 there should be a vertical negation stroke attached to the stroke preceding the first occurrence of "$h(y)$"; on page 39 an unreadable broken "c" has been left uncorrected.

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PREFACE

In apprehending a scientific truth we pass, as a rule, through various degrees of certitude. Perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established. Hence we can inquire, on the one hand, how we have gradually arrived at a given proposition and, on the other, how we can finally provide it with the most secure foundation. The first question may have to be answered differently for different persons; the second is more definite, and the answer to it is connected with the inner nature of the proposition considered. The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests. Accordingly, we divide all truths that require justification into two kinds, those for which the proof can be carried out purely by means of logic and those for which it must be supported by facts of experience. But that a proposition is of the first kind is surely compatible with the fact that it could nevertheless not have come to consciousness in a human mind without any activity of the senses.[1] Hence it is not the psychological genesis but the best method of proof that is at the basis of the classification. Now, when I came to consider the question to which of these two kinds the judgments of arithmetic belong, I first had to ascertain how far one could proceed in arithmetic by means of inferences alone, with the sole support of those laws of thought that transcend all particulars. My initial step was to attempt to reduce the concept of ordering in a sequence to that of *logical* consequence, so as to proceed from there to the concept of number. To prevent anything intuitive [[Anschauliches]] from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an

[1] Since without sensory experience no mental development is possible in the beings known to us, that holds of all judgments.

obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated. That is why I decided to forgo expressing anything that is without significance for the *inferential sequence*. In § 3 I called what alone mattered to me the *conceptual content* ⟦*begrifflichen Inhalt*⟧. Hence this definition must always be kept in mind if one wishes to gain a proper understanding of what my formula language is. That, too, is what led me to the name "Begriffsschrift". Since I confined myself for the time being to expressing relations that are independent of the particular characteristics of objects, I was also able to use the expression "formula language for pure thought". That it is modeled upon the formula language of arithmetic, as I indicated in the title, has to do with fundamental ideas rather than with details of execution. Any effort to create an artificial similarity by regarding a concept as the sum of its marks ⟦Merkmale⟧ was entirely alien to my thought. The most immediate point of contact between my formula language and that of arithmetic is the way in which letters are employed.

I believe that I can best make the relation of my ideography to ordinary language ⟦Sprache des Lebens⟧ clear if I compare it to that which the microscope has to the eye. Because of the range of its possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. Considered as an optical instrument, to be sure, it exhibits many imperfections, which ordinarily remain unnoticed only on account of its intimate connection with our mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others.

This ideography, likewise, is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others. If it answers to these purposes in some degree, one should not mind the fact that there are no new truths in my work. I would console myself on this point with the realization that a development of method, too, furthers science. Bacon, after all, thought it better to invent a means by which everything could easily be discovered than to discover particular truths, and all great steps of scientific progress in recent times have had their origin in an improvement of method.

Leibniz, too, recognized—and perhaps overrated—the advantages of an adequate system of notation. His idea of a universal characteristic, of a *calculus philosophicus* or *ratiocinator*,[2] was so gigantic that the attempt to realize it could not go beyond the bare preliminaries. The enthusiasm that seized its originator when he contemplated the immense increase in the intellectual power of mankind that a system of notation directly appropriate to objects themselves would bring about led him to underestimate the difficulties that stand in the way of such an enterprise. But, even if this worthy goal cannot be reached in one leap, we need not despair of a slow, step-by-step approximation. When a problem appears to be unsolvable in its full generality, one should

[2] On that point see *Trendelenburg 1867* ⟦pp. 1–47, *Ueber Leibnizens Entwurf einer allgemeinen Charakteristik*⟧.

temporarily restrict it; perhaps it can then be conquered by a gradual advance. It is possible to view the signs of arithmetic, geometry, and chemistry as realizations, for specific fields, of Leibniz's idea. The ideography proposed here adds a new one to these fields, indeed the central one, which borders on all the others. If we take our departure from there, we can with the greatest expectation of success proceed to fill the gaps in the existing formula languages, connect their hitherto separated fields into a single domain, and extend this domain to include fields that up to now have lacked such a language.[3]

I am confident that my ideography can be successfully used wherever special value must be placed on the validity of proofs, as for example when the foundations of the differential and integral calculus are established.

It seems to me to be easier still to extend the domain of this formula language to include geometry. We would only have to add a few signs for the intuitive relations that occur there. In this way we would obtain a kind of *analysis situs*.

The transition to the pure theory of motion and then to mechanics and physics could follow at this point. The latter two fields, in which besides rational necessity ⟦Denknothwendigkeit⟧ empirical necessity ⟦Naturnothwendigkeit⟧ asserts itself, are the first for which we can predict a further development of the notation as knowledge progresses. That is no reason, however, for waiting until such progress appears to have become impossible.

If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts and by freeing thought from that with which only the means of expression of ordinary language, constituted as they are, saddle it, then my ideography, further developed for these purposes, can become a useful tool for the philosopher. To be sure, it too will fail to reproduce ideas in a pure form, and this is probably inevitable when ideas are represented by concrete means; but, on the one hand, we can restrict the discrepancies to those that are unavoidable and harmless, and, on the other, the fact that they are of a completely different kind from those peculiar to ordinary language already affords protection against the specific influence that a particular means of expression might exercise.

The mere invention of this ideography has, it seems to me, advanced logic. I hope that logicians, if they do not allow themselves to be frightened off by an initial impression of strangeness, will not withhold their assent from the innovations that, by a necessity inherent in the subject matter itself, I was driven to make. These deviations from what is traditional find their justification in the fact that logic has hitherto always followed ordinary language and grammar too closely. In particular, I believe that the replacement of the concepts *subject* and *predicate* by *argument* and *function*, respectively, will stand the test of time. It is easy to see how regarding a content as a function of an argument leads to the formation of concepts. Furthermore, the demonstration of the connection between the meanings of the words *if*, *and*, *not*, *or*, *there is*, *some*, *all*, and so forth, deserves attention.

Only the following point still requires special mention. The restriction, in § 6, to a

[3] ⟦On that point see *Frege 1879a.*⟧

comprehensive domain of pure thought in general. I therefore divide all signs that I use into *those by which we may understand different objects* and *those that have a completely determinate meaning.* The former are *letters* and they will serve chiefly to express *generality.* But, no matter how indeterminate the meaning of a letter, we must insist that throughout a given context the letter *retain* the meaning once given to it.

Judgment

§ 2. A judgment will always be expressed by means of the sign

$$\vdash\!\!\!-\!\!-\,,$$

which stands to the left of the sign, or the combination of signs, indicating the content of the judgment. If we *omit* the small vertical stroke at the left end of the horizontal one, the judgment will be transformed into a *mere combination of ideas* [*Vorstellungsverbindung*],[6] of which the writer does not state whether he acknowledges it to be true or not. For example, let

$$\vdash\!\!\!-\!\!-\!\!- A$$

stand for ⟦bedeute⟧ the judgment "Opposite magnetic poles attract each other";[7] then

$$-\!\!\!-\!\!\!- A$$

will not express ⟦ausdrücken⟧ this judgment;[8] it is to produce in the reader merely the idea of the mutual attraction of opposite magnetic poles, say in order to derive consequences from it and to test by means of these whether the thought is correct. When the vertical stroke is omitted, we express ourselves *paraphrastically*, using the words "the circumstance that" or "the proposition that".[9]

Not every content becomes a judgment when $\vdash\!\!\!-\!\!-$ is written before its sign; for

[6] ⟦Footnote by Jourdain (*1912*, p. 242):
"For this word I now simply say 'Gedanke'. The word 'Vorstellungsinhalt' is used now in a psychological, now in a logical sense. Since this creates obscurities, I think it best not to use this word at all in logic. We must be able to express a thought without affirming that it is true. If we want to characterize a thought as false, we must first express it without affirming it, then negate it, and affirm as true the thought thus obtained. We cannot correctly express a hypothetical connection between thoughts at all if we cannot express thoughts without affirming them, for in the hypothetical connection neither the thought appearing as antecedent nor that appearing as consequent is affirmed." [Frege, 1910.]⟧

[7] I use Greek letters as abbreviations, and to each of these letters the reader should attach an appropriate meaning when I do not expressly give them a definition. ⟦The "A" that Frege is now using is a capital alpha.⟧

[8] ⟦Jourdain had originally translated "bedeuten" by "signify", and Frege wrote (see *Jourdain 1912*, p. 242):
"Here we must notice the words 'signify' and 'express'. The former seems to correspond to 'bezeichnen' or 'bedeuten', the latter to 'ausdrücken'. According to the way of speaking I adopted I say 'A proposition expresses a thought and signifies its truth value'. Of a judgment we cannot properly say either that it signifies or that it is expressed. We do, to be sure, have a thought in the judgment, and that can be expressed; but we have more, namely, the recognition of the truth of this thought."⟧

[9] ⟦Footnote by Jourdain (*1912*, p. 243):
"Instead of 'circumstance' and 'proposition' I would simply say 'thought'. Instead of 'beurtheilbarer Inhalt' we can also say 'Gedanke'." [Frege, 1910.]⟧

example, the idea "house" does not. We therefore distinguish contents that *can become a judgment* from those that *cannot*.[10]

The horizontal stroke that is part of the sign ├── *combines the signs that follow it into a totality, and the affirmation expressed by the vertical stroke at the left end of the horizontal one refers to this totality.* Let us call the horizontal stroke the *content stroke* and the vertical stroke the *judgment stroke*. The content stroke will in general serve to relate any sign to the totality of the signs that follow the stroke. *Whatever follows the content stroke must have a content that can become a judgment*.

§ 3. A distinction between *subject* and *predicate* does *not occur* in my way of representing a judgment. In order to justify this I remark that the contents of two judgments may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgments, always follow also from the second, when it is combined with these same judgments, [[and conversely,]] or this is not the case. The two propositions "The Greeks defeated the Persians at Plataea" and "The Persians were defeated by the Greeks at Plataea" differ in the first way. Even if one can detect a slight difference in meaning, the agreement outweighs it. Now I call that part of the content that is the *same* in both the *conceptual content*. Since *it alone* is of significance for our ideography, we need not introduce any distinction between propositions having the same conceptual content. If one says of the subject that it "is the concept with which the judgment is concerned", this is equally true of the object. We can therefore only say that the subject "is the concept with which the judgment is chiefly concerned". In ordinary language, the place of the subject in the sequence of words has the significance of a *distinguished* place, where we put that to which we wish especially to direct the attention of the listener (see also § 9). This may, for example, have the purpose of pointing out a certain relation of the given judgment to others and thereby making it easier for the listener to grasp the entire context. Now, all those peculiarities of ordinary language that result only from the interaction of speaker and listener—as when, for example, the speaker takes the expectations of the listener into account and seeks to put them on the right track even before the complete sentence is enunciated—have nothing that answers to them in my formula language, since in a judgment I consider only that which influences its *possible consequences*. Everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated; *nothing is left to guesswork*. In this I faithfully follow the example of the formula language of mathematics, a language to which one would do violence if he were to distinguish between subject and predicate in it. We can imagine a language in which the proposition "Archimedes perished at the capture of Syracuse" would be expressed thus: "The violent death of Archimedes at the capture of Syracuse is a fact". To be sure, one can distinguish between subject and predicate here, too, if one wishes to do so, but the subject contains the whole content, and the predicate serves only to turn the content into a judgment. *Such a*

[10] On the other hand, the circumstance that there are houses, or that there is a house (see § 12 [[footnote 15]]), is a content that can become a judgment. But the idea "house" is only a part of it. In the proposition "The house of Priam was made of wood" we could not put "circumstance that there is a house" in place of "house". For a different kind of example of a content that cannot become a judgment see the passage following formula (81).

[[In German Frege's distinction is between "beurtheilbare" and "unbeurtheilbare" contents. Jourdain uses the words "judicable" and "nonjudicable".]]