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Admissible rules in the implication–negation fragment of intuitionistic logic

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ABSTRACT

Uniform infinite bases are defined for the single-conclusion and multiple-conclusion admissible rules of the implication–negation fragments of intuitionistic logic IPC and its consistent axiomatic extensions (intermediate logics). A Kripke semantics characterization is given for the (hereditarily) structurally complete implication–negation fragments of intermediate logics, and it is shown that the admissible rules of this fragment of IPC form a PSPACE-complete set and have no finite basis.

1. Introduction

Following Lorenzen [17], a rule is said to be *admissible* for a logic (understood as a finitary structural consequence relation) if it can be added to a proof system for the logic without producing any new theorems. While the admissible rules of classical propositional logic CPC are also *derivable* – that is, CPC is *structurally complete* – this is not the case for non-classical (modal, many-valued, substructural, intermediate) logics in general (see, e.g., [26,22,4]). In particular, the study of admissible rules was stimulated by the discovery of admissible but underivable rules of intuitionistic propositional logic IPC such as the independence of premises rule:

$$\neg p \to (q \lor r) \, / \, (\neg p \to q) \lor (\neg p \to r).$$

The decidability of the set of admissible rules of IPC, posed as an open problem by Friedman in [6], was answered positively by Rybakov, who demonstrated also that this set has no finite *basis* (understood as a set of admissible rules that added to IPC produces all admissible rules) [26]. Nevertheless, following a conjecture by de Jongh and Visser, Iemhoff [9] and Rozière [25] established independently that an infinite basis is formed by the family of "Visser rules" ($n = 2, 3, \ldots$):

$$\left(\bigwedge_{i=1}^{n} (q_i \to p_i) \to (q_{n+1} \lor q_{n+2}) \right) \lor r \Bigg/ \bigvee_{j=1}^{n+2} \left(\bigwedge_{i=1}^{n} (q_i \to p_i) \to q_j \right) \lor r.$$

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More generally, the work of Rybakov [26] and Ghilardi [7,8] has led to a reasonably comprehensive understanding of structural completeness and admissible rules for broad classes of intermediate and modal logics. Kripke frame based characterizations of *hereditarily structurally complete* (i.e., each extension of the logic is structurally complete) intermediate logics and transitive modal logics have been obtained by Citkin and Rybakov [5,26]. Bases have been provided for certain intermediate logics by Iemhoff [10] and for transitive modal logics by Jeřábek [13], and Gentzen-style proof systems have been developed for these logics by Iemhoff and Metcalfe [11,12]. Note, moreover, that in these cases, admissibility is characterized in the wider setting of multiple-conclusion rules, where, as the name suggests, many conclusions as well as many premises are permitted. A paradigmatic example of a multiple-conclusion rule admissible in intuitionistic logic but not classical logic is the disjunction property, which may be formulated as

$$p \vee q \,/\, p, q.$$

For other families of non-classical logics, much less is known, but structural completeness for substructural logics has been investigated by Olson et al. [22] and for fuzzy logics by the current authors [4], and bases have been provided for the (multiple-conclusion) admissible rules of Łukasiewicz logics by Jeřábek [14,15] and the logic R-Mingle by Metcalfe [19].

Hereditary structural completeness for the implicational fragment of IPC was established by Prucnal [23], and the same proof method extends to the implication–conjunction and implication–conjunction–negation fragments [20]. Mints demonstrated hereditary structural completeness for implicationless fragments of IPC and showed moreover that any admissible underivable rule of IPC must contain both implication and disjunction [21]. Curiously, however, as observed by Wroński [28], the *implication–negation fragment* (equivalently, the implication–falsity fragment) – the logic of bounded BCKW-algebras – is not structurally complete. Consider, e.g., the following rule:

$$((\neg\neg p \to p) \to r), ((\neg\neg q \to q) \to r), (p \to \neg q) \,/\, r.$$

This rule is not derivable in IPC and therefore not in any of its fragments. The rule is also not admissible in IPC. However, it *is* admissible in the implication–negation fragment of this logic.

Hence the questions arise: Do there exist other admissible underivable rules for this fragment of a similar or quite different form? Do these admissible rules admit an elegant finite or infinite basis? Do they form a decidable set and if so, what is its complexity? What is the unification type of this fragment? This paper answers these questions as follows:

- Elegant bases consisting of uniform infinite sequences of rules, similar to Wroński's example, are provided for the single-conclusion and multiple-conclusion admissible rules of the implication–negation fragment not only of IPC but also of any intermediate logic (Theorems 3.5 and 3.6).
- The admissible rules of this fragment of IPC are shown to form a PSPACE-complete set (Theorem 4.3).
- A Kripke frame characterization is given of the (hereditarily) structurally complete intermediate logics with respect to the implication–negation fragment (Theorem 5.3) and used to show the lack of a finite basis for this fragment of IPC (Theorem 5.5).
- It is shown that the unification type of the implication–negation fragment of any intermediate logic properly included in classical logic is finitary and not unitary (Theorem 6.1).

2. Preliminaries

We begin by fixing some basic notation and definitions, in particular, for dealing with multiple-conclusion and single-conclusion consequence relations, derivable and admissible rules, and, crucially, the concept of projectivity for a logic.

2.1. Consequence relations

The notions of a *propositional language* $\mathcal{L}$ (a set of *connectives* with specified finite arities) and set of *$\mathcal{L}$-formulas* $\mathrm{Fm}_{\mathcal{L}}$ over a fixed countably infinite set of variables $p, q, r, \ldots$ are defined as usual, denoting formulas by φ, ψ, χ and finite sets of formulas by $\Gamma, \Delta, \Pi, \Sigma$. An *$\mathcal{L}$-substitution* σ is then an endomorphism on the formula algebra $\mathbf{Fm}_{\mathcal{L}}$, writing $\sigma(\Gamma)$ for $\{\sigma\varphi \mid \varphi \in \Gamma\}$.

A *rule* for $\mathcal{L}$ is an ordered pair (Γ, Δ), written as $\Gamma \,/\, \Delta$, where $\Gamma \cup \Delta$ is a finite subset of $\mathrm{Fm}_{\mathcal{L}}$, called *single-conclusion* if $|\Delta| = 1$ and *multiple-conclusion* in general. We write '$\Gamma \,/\, \varphi$', 'Γ, Δ', and 'Γ, φ' for, respectively, '$\Gamma \,/\, \{\varphi\}$', '$\Gamma \cup \Delta$', and '$\Gamma \cup \{\varphi\}$'. A *(finitary structural) multiple-conclusion consequence relation* on $\mathbf{Fm}_{\mathcal{L}}$, or *m-logic* for short, is then a set L of rules (writing $\Gamma \vdash_{\mathrm{L}} \Delta$ instead of $(\Gamma, \Delta) \in \mathrm{L}$) satisfying for all finite subsets $\Gamma, \Gamma', \Delta, \Delta'$ of $\mathrm{Fm}_{\mathcal{L}}$ and formulas $\varphi \in \mathrm{Fm}_{\mathcal{L}}$:

1. $\varphi \vdash_{\mathrm{L}} \varphi$,
2. if $\Gamma \vdash_{\mathrm{L}} \Delta$, then $\Gamma, \Gamma' \vdash_{\mathrm{L}} \Delta', \Delta$,
3. if $\Gamma, \varphi \vdash_{\mathrm{L}} \Delta$, and $\Gamma' \vdash_{\mathrm{L}} \varphi, \Delta'$, then $\Gamma, \Gamma' \vdash_{\mathrm{L}} \Delta', \Delta$,
4. if $\Gamma \vdash_{\mathrm{L}} \Delta$, then $\sigma(\Gamma) \vdash_{\mathrm{L}} \sigma(\Delta)$ for each $\mathcal{L}$-substitution σ.

A *(finitary structural) consequence relation*, or *logic* for short, is a set L of single-conclusion rules satisfying (the corresponding single-conclusion variants of) parts 1–4 of the definition of m-logic.[1] A *theorem* of an (m-)logic L is a formula φ such that $\emptyset \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

In this paper, our primary interest lies with logics not m-logics, the latter being essentially a technical tool useful for simplifying proofs and obtaining more uniform presentations. Nevertheless, it will be helpful to think of each logic L as determining an m-logic:

$$\mathrm{L_m} = \{\Gamma \ / \ \Delta \mid (\exists \varphi \in \Delta)(\Gamma \vdash_{\mathrm{L}} \varphi)\}.$$

Clearly $\mathrm{L_m}$ is an m-logic. Let us also define the single-conclusion fragment of an m-logic L as

$$\mathrm{L_s} = \{(\Gamma \ / \ \Delta) \in \mathrm{L} \mid |\Delta| = 1\}.$$

Then $(\mathrm{L_m})_\mathrm{s}$ is L. On the other hand, for an m-logic L, in general $(\mathrm{L_s})_\mathrm{m} \neq \mathrm{L}$. Consider, e.g., an m-logic L defined by $\Gamma \vdash_{\mathrm{L}} \Delta$ iff $\Gamma \vdash_{\mathrm{CPC}} \bigvee \Delta$. Then $\vdash_{\mathrm{L}} \{p, \neg p\}$, but $\vdash_{(\mathrm{L_s})_\mathrm{m}} \{p, \neg p\}$ iff $\vdash_{\mathrm{L_s}} p$ or $\vdash_{\mathrm{L_s}} \neg p$, i.e., iff $\vdash_{\mathrm{CPC}} p$ or $\vdash_{\mathrm{CPC}} \neg p$, which does not hold.

2.2. Derivable and admissible rules

A rule $\Gamma \ / \ \Delta$ is said to be *derivable* in a logic or m-logic L if $\Gamma \vdash_{\mathrm{L}} \Delta$, and *admissible* in L, written $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$, if for each substitution σ: whenever $\vdash_{\mathrm{L}} \sigma\varphi$ for all $\varphi \in \Gamma$, also $\vdash_{\mathrm{L}} \sigma\psi$ for some $\psi \in \Delta$. Observe that $\mathrel{|\!\!\sim}_{\mathrm{L}}$ is itself an m-logic, even in the case where L is a logic.

Example 2.1. Although single-conclusion rules are usually of greater interest, multiple-conclusion rules can be useful for expressing important properties of logics. For example, the *disjunction property* can be formulated as $p \vee q \ / \ p, q$. Observe that this rule is IPC-admissible but not CPC-admissible: just consider the substitution $\sigma p = p$ and $\sigma q = \neg p$.

A logic L is said to be *structurally complete* if all logics $\mathrm{L}' \supsetneq \mathrm{L}$ in the same language have new theorems, and *hereditarily structurally complete* if all logics $\mathrm{L}' \supseteq \mathrm{L}$ in the same language are structurally complete. It is easily proved (see, e.g., [22]) that a logic L is structurally complete if and only if L coincides with the single-conclusion fragment of $\mathrel{|\!\!\sim}_{\mathrm{L}}$. Examples of (hereditarily) structurally complete logics include CPC, Gödel logic G, and the implicational fragment of IPC. More details, including algebraic characterizations of these notions, may be found in [22,4].

For a logic L that is not structurally complete, we are interested in axiomatizing the admissible rules of L by adding a suitable set of rules as a "basis". More generally, we define what it means for a set of rules to be a basis for one logic or m-logic over another. Let L be an m-logic (logic) and B a set of (single-conclusion) rules. Then L^B denotes the smallest m-logic (logic) containing $\mathrm{L} \cup B$ and B is called a *basis* for L^B over L. In particular, our aim will be to find for a logic L: (a) a basis for $\mathrel{|\!\!\sim}_{\mathrm{L}}$ over $\mathrm{L_m}$, and (b) a basis for $(\mathrel{|\!\!\sim}_{\mathrm{L}})_s$ over L.

2.3. Projectivity

Although a logic may not be structurally complete, there may be well-behaved sets of formulas such that for rules whose premises form such a set, admissibility coincides with derivability. Let us fix L as a logic based on a language $\mathcal{L}$ containing a binary connective $\to$ for which modus ponens is derivable ($\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$). Generalizing Ghilardi [7,8] slightly, $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is called L-*projective* if there exists an $\mathcal{L}$-substitution σ, called an L-*projective unifier* for Γ, such that (i) σ is an L-unifier for Γ, namely $\vdash_{\mathrm{L}} \sigma\varphi$ for all $\varphi \in \Gamma$, and (ii) $\Gamma \vdash_{\mathrm{L}} \sigma\psi \to \psi$ and $\Gamma \vdash_{\mathrm{L}} \psi \to \sigma\psi$ for all $\psi \in \mathrm{Fm}_{\mathcal{L}}$. (We also say that $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ is L-projective if $\{\varphi\}$ is L-projective to conform with Ghilardi's definition.) Moreover, such a σ is also a *most general* L-*unifier* for Γ in the sense that for any other L-unifier σ_1 for Γ, there exists an $\mathcal{L}$-substitution σ_2 such that $\sigma_2\sigma = \sigma_1$.

Example 2.2. Notice that in IPC, any formula of the form $p \to \varphi$ or $\varphi \to p$ is IPC-projective, with corresponding IPC-projective unifier $\sigma q = (p \to \varphi) \wedge q$ or $\sigma q = (p \to \varphi) \to q$, respectively. On the other hand, formulas such as $\neg p \to (q \vee r)$ and $\neg p \vee \neg\neg p$ are not IPC-projective.

Lemma 2.3. *If $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is* L*-projective, then:* (i) $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$ *iff* $\Gamma \vdash_{\mathrm{L_m}} \Delta$*;* (ii) $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \varphi$ *iff* $\Gamma \vdash_{\mathrm{L}} \varphi$*.*

Proof. Notice that (ii) immediately follows from (i). For the right-to-left direction of (i), observe that if $\Gamma \vdash_{\mathrm{L_m}} \Delta$, then $\Gamma \vdash_{\mathrm{L}} \varphi$ for some $\varphi \in \Delta$. So $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for any $\mathcal{L}$-substitution σ, and if $\vdash_{\mathrm{L}} \sigma\psi$ for each $\psi \in \Gamma$, then $\vdash_{\mathrm{L}} \sigma\varphi$. That is, $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$. For the other direction, let σ be an L-projective unifier of Γ. If $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$, then $\vdash_{\mathrm{L}} \sigma\psi$ for some $\psi \in \Delta$. Since σ is an L-projective unifier, $\Gamma \vdash_{\mathrm{L}} \sigma\psi \to \psi$. Hence by modus ponens, $\Gamma \vdash_{\mathrm{L}} \psi$ and we get $\Gamma \vdash_{\mathrm{L_m}} \Delta$ as required. $\square$

It can be checked, following [23], that for any intermediate logic L and finite set of implicational formulas $\{\varphi_1, \ldots, \varphi_n\}$, the substitution $\sigma p = \varphi_1 \to (\varphi_2 \to (\cdots \to (\varphi_n \to p) \cdots)$ is an L-projective unifier. So by Lemma 2.3, the implicational fragment of any intermediate logic is (hereditarily) structurally complete. This reasoning extends also to the implication–conjunction and implication–conjunction–negation fragments [20], but may no longer hold for the full logic. In particular, although any formula of the form $p \to \varphi$ is IPC-projective, in general, formulas such as $\varphi \vee \psi$ are not, and indeed IPC is not structurally complete. Most relevantly for the current paper, formulas of the form $\neg\varphi$ are generally not projective for the implication–negation fragment of IPC or many other intermediate logics.

[1] We remark that this definition of a logic does not quite match the usual Tarski-style definition which allows the set of formulas Γ occurring in $\Gamma \vdash_{\mathrm{L}} \varphi$ to be infinite. Nevertheless, every finitary structural consequence relation in the usual sense determines a unique logic in our sense and vice versa.

3. Bases for intermediate logics

For convenience, let us assume for the remainder of this paper that L is a consistent axiomatic extension of the implication–negation fragment of intuitionistic logic IPC (ensuring, e.g., that L has the deduction theorem and is contained in classical logic CPC). Note that our convention includes (but is not necessarily limited to) the implication–negation fragment of any intermediate logic (axiomatic extension of IPC).[2]

The basic connectives of L are taken to be $\to$ and $\bot$, defining $\neg\varphi =_{\text{def}} \varphi \to \bot$ and $\top =_{\text{def}} \bot \to \bot$. We abbreviate $\varphi_1 \to (\varphi_2 \to (\cdots \to (\varphi_n \to \psi)\cdots)$ by $\varphi_1 \to \varphi_2 \to \cdots \to \varphi_n \to \psi$ or $\vec{\varphi} \to \psi$ and, where appropriate, treat $\vec{\varphi}$ as a set in our proofs, making no distinction syntactically between formulas with permuted antecedents (such as $\varphi_1 \to \varphi_2 \to \psi$ and $\varphi_2 \to \varphi_1 \to \psi$) or with multiple occurrences of the same antecedent (such as $\varphi \to \vec{\varphi} \to \varphi \to \psi$ and $\varphi \to \vec{\varphi} \to \psi$). For $\vec{\varphi} = \emptyset$, we understand $\vec{\varphi} \to \psi$ to be the formula ψ. We use $\Gamma, \Pi, \Delta, \Sigma$ without further comment to denote finite sets of formulas and p, q, r to denote propositional variables. Since by the Glivenko theorem and the fact that L is contained in CPC, a set of formulas is L-consistent if and only if $\Gamma \nvdash_{\text{CPC}} \bot$, we drop the prefix and speak just of consistency.

3.1. The Wroński rules

Bases for the multiple-conclusion and single-conclusion admissible rules of L will consist of sets of the following "Wroński rules" ($n \in \mathbb{N}$):

(W_n) $\quad (\vec{p} \to \bot) \,/\, (\neg\neg p_1 \to p_1), \ldots, (\neg\neg p_n \to p_n)$

(W'_n) $\quad (\vec{p} \to \bot), ((\neg\neg p_1 \to p_1) \to q), \ldots, ((\neg\neg p_n \to p_n) \to q) \,/\, q.$

Note that in the case of $n = 0$ (useful for technical reasons) (W_0) is $\bot \,/\, \emptyset$ and is L-admissible but not L-derivable, and (W'_0) is $\bot \,/\, q$ and is both L-admissible and L-derivable. Note also that (W'_2) is the example of Wroński mentioned in the introduction.

The single-conclusion rules (W'_n) for $n \geq 2$ are not derivable in IPC (see Lemma 5.1) but are derivable in stronger intermediate logics such as Gödel logic (IPC + $(p \to q) \vee (q \to p)$) and De Morgan (Jankov) logic (IPC + $\neg p \vee \neg\neg p$). Nevertheless, in all cases:

Lemma 3.1. *(W_n) and (W'_n) are* L*-admissible for all $n \in \mathbb{N}$.*[3]

Proof. Suppose that $\vdash_{\text{L}} \sigma(\vec{p} \to \bot)$, noting in particular that when $n = 0$, this is not possible. Then σp_i must be of the form $\vec{\varphi} \to \bot$ for some $i \in \{1, \ldots, n\}$; otherwise, the substitution $\sigma' q = \top$ for each variable q gives $\vdash_{\text{L}} \sigma'\sigma(\vec{p} \to \bot)$ and therefore $\vdash_{\text{L}} \top \to \bot$, a contradiction. Hence, since $\vdash_{\text{L}} \neg\neg(\vec{\varphi} \to \bot) \to (\vec{\varphi} \to \bot)$ we obtain: $\vdash_{\text{L}} \sigma(\neg\neg p_i \to p_i)$. So (W_n) is L-admissible. Moreover, since if $\vdash_{\text{L}} \sigma((\neg\neg p_i \to p_i) \to q)$, then $\vdash_{\text{L}} \sigma q$, also (W'_n) is L-admissible. □

Observe on the other hand that these rules may not be admissible in fragments of an intermediate logic containing $\wedge$ or $\vee$ as well as $\to$ and $\bot$. In particular, for IPC, let $\sigma p_1 = p \wedge \neg q$ and $\sigma p_2 = q$. Then $\vdash_{\text{IPC}} \sigma(p_1 \to p_2 \to \bot)$ but $\nvdash_{\text{IPC}} \sigma(\neg\neg p_1 \to p_1)$ and $\nvdash_{\text{IPC}} \sigma(\neg\neg p_2 \to p_2)$.

3.2. A multiple-conclusion basis

We define the following set of multiple-conclusion Wroński rules:

$$W = \{(W_n) \mid n \in \mathbb{N}\}.$$

Our aim is to show that $\Gamma \mathrel{|\!\!\sim}_{\text{L}} \Delta$ implies $\Gamma \vdash_{\text{L}^W_{\text{m}}} \Delta$ (the reverse direction follows from Lemma 3.1). The first step of our strategy will be to "reduce" the question of the admissibility of any rule to the admissibility of rules of a certain basic form. Let us call a formula having one of the following forms *simple*:

(i) $\vec{p} \to \bot$,
(ii) $\vec{\psi} \to r$ where each member of $\vec{\psi}$ is of the form $p \to q$ or p.

The next lemma formalizes this "reduction" idea; its proof is based on replacing non-simple formulas by formulas which are, in a sense, "more simple".

Lemma 3.2. *For any finite set of formulas Γ, there exists a finite set of* simple *formulas Π such that for each finite set of formulas Δ:*

1. $\Gamma \vdash_{\text{L}^W_{\text{m}}} \Delta$ *iff* $\Pi \vdash_{\text{L}^W_{\text{m}}} \Delta$,
2. $\Gamma \mathrel{|\!\!\sim}_{\text{L}} \Delta$ *iff* $\Pi \mathrel{|\!\!\sim}_{\text{L}} \Delta$.

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2 This follows from the fact that any such fragment has the classical deduction theorem and an extension of the implication–negation fragment of IPC has the classical deduction theorem iff it is an *axiomatic* extension (a folklore result; for an explicit formulation see, e.g., [3, Corollary 8]).

3 This statement holds more widely in fact, e.g., for the implication–negation fragment of any consistent axiomatic extension of the full Lambek calculus with exchange and weakening FL_{ew} (equivalently, affine multiplicative additive intuitionistic linear logic or monoidal logic).

Proof. Define the *complexity* of a formula φ to be the number of occurrences of $\to$ and $\bot$ in φ, and let $mc(\Gamma)$ be the multiset of complexities of the formulas occurring in Γ. We prove the claim by induction on $mc(\Gamma)$ using the standard multiset ordering $<_m$: the transitive closure of $<^1_m$ defined for finite sets of natural numbers P, Q by $P <^1_m Q$ iff P can be obtained from Q by replacing an element x by $y_1, \ldots, y_n$ where $y_i < x$ for $i = 1, \ldots, n$.

Suppose that there is a formula φ in Γ that is not simple. If φ is of the form $\bot \to \psi$ (possibly permuting antecedents), then we remove φ from Γ and the result follows using the induction hypothesis. Now suppose that φ is of the form $(\psi_1 \to \psi_2) \to \chi$ (possibly permuting antecedents) where either ψ_1 or ψ_2 is not a variable. We obtain a set of formulas Γ' with $mc(\Gamma') <_m mc(\Gamma)$ by replacing φ with the formulas $(p \to q) \to \chi, p \to \psi_1$, and $\psi_2 \to q$ where p and q do not occur in Γ, φ, or Δ. The result follows from the induction hypothesis and the derivabilities:

(1) $(p \to q) \to \chi, p \to \psi_1, \psi_2 \to q \vdash_{\mathrm{L}} (\psi_1 \to \psi_2) \to \chi$,
(2) $\Gamma \setminus \{(\psi_1 \to \psi_2) \to \chi\}, (p \to q) \to \chi, p \to \psi_1, \psi_2 \to q \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$.

Just note that since p and q are new variables, using (2), we obtain $\Gamma, \psi_1 \to \psi_1, \psi_2 \to \psi_2 \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$, i.e., $\Gamma \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$ as required. Finally, the only remaining possibility is that φ is of the form $(p \to q) \to \vec{\psi} \to \bot$ (possibly permuting antecedents). We obtain a set of formulas Γ' with $mc(\Gamma') <_m mc(\Gamma)$ by replacing φ with the formulas $q \to \vec{\psi} \to \bot$ and $(p \to q) \to \vec{\psi} \to p$. The result follows from the induction hypothesis and the derivabilities:

(3) $q \to \vec{\psi} \to \bot, (p \to q) \to \vec{\psi} \to p \vdash_{\mathrm{L}} (p \to q) \to \vec{\psi} \to \bot$,
(4) $(p \to q) \to \vec{\psi} \to \bot \vdash_{\mathrm{L}} q \to \vec{\psi} \to \bot$,
(5) $(p \to q) \to \vec{\psi} \to \bot \vdash_{\mathrm{L}} (p \to q) \to \vec{\psi} \to p$. □

Hence to show that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ implies $\Gamma \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$, we can use this lemma to construct the finite set of simple formulas Π such that $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ and it then suffices to show that $\Pi \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$. Our strategy will be to construct a finite set Ψ_Π of finite sets of formulas containing Π such that $\Pi \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$ whenever $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$, and then to reduce the admissibility problem $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ to the derivability problem $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$. Roughly speaking, Ψ_Π will contain all possible applications of the rules from W to formulas in Π; i.e., making use of the deduction theorem, we consider sets of variables X such that

$$\Pi \cup X \vdash_{\mathrm{L}} \bot$$

and obtain new sets of formulas by adding for each such X a formula $\neg\neg p \to p$ for some $p \in X$.

Let us elaborate these ideas in detail. By $\mathrm{Var}(\Pi)$ we denote the set of variables occurring in Π. We enumerate the sets of variables $X \subseteq \mathrm{Var}(\Pi)$ such that $\Pi \cup X \vdash_{\mathrm{L}} \bot$ as $X_1, \ldots, X_n$ and define the following sequence:

1. $\Psi_0 = \{\emptyset\}$,
2. $\Psi_i = \{\Sigma \cup \{\neg\neg p \to p\} \mid \Sigma \in \Psi_{i-1} \text{ and } p \in X_i\}$ for $i = 1, \ldots, n$.

Let $\Psi_\Pi = \{\Sigma \cup \Pi \mid \Sigma \in \Psi_n\}$. Note that if Π is inconsistent, then $X_i = \emptyset$ for some $i \in \{1, \ldots, n\}$ and it follows that $\Psi_j = \emptyset$ for $i \le j \le n$, and so $\Psi_\Pi = \emptyset$.

Lemma 3.3. *Let Π be a finite set of simple formulas. If $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$, then $\Pi \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$.*

Proof. Note as a special case that if Π is inconsistent, then the result follows immediately using the rule (W_0). More generally, we prove by induction on $i = 0, \ldots, n$, that $\Pi' \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$ for all $\Pi' \in \{\Sigma \cup \Pi \mid \Sigma \in \Psi_i\}$ implies $\Pi \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$. The base case is immediate. For the induction step suppose that $\Pi \cup \Sigma \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$ for all $\Sigma \in \Psi_i$. Consider some $\Sigma' \in \Psi_{i-1}$. By construction, $\Pi \vdash_{\mathrm{L}} X_i \to \bot$ and $\Pi \cup \Sigma' \cup \{\neg\neg p \to p\} \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$ for each $p \in X_i$. But $X_i \to \bot \vdash_{\mathrm{L}^W_{\mathrm{m}}} \{\neg\neg p \to p \mid p \in X_i\}$ is an instance of a rule of W. Hence by transitivity, $\Pi \cup \Sigma' \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$. So by the induction hypothesis, $\Pi \vdash_{\mathrm{L}^W_{\mathrm{m}}} \Delta$ as required. □

It remains then to show that $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$ whenever $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ for any finite set of simple formulas Π. The crucial step here will be to establish for each $\Pi' \in \Psi_\Pi$ that there exists an L-projective set of formulas $\vec{\varphi}$ such that $\Pi \subseteq \vec{\varphi} \subseteq \Pi'$. It then follows that $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ (trivially) implies $\vec{\varphi} \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ and so by Lemma 2.3 we obtain $\vec{\varphi} \vdash_{\mathrm{L_m}} \Delta$ and hence (trivially) also $\Pi' \vdash_{\mathrm{L_m}} \Delta$.

Lemma 3.4. *Let Π be a finite set of simple formulas. If $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$, then $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$.*

Proof. As mentioned above, it suffices to show that for each $\Pi' \in \Psi_\Pi$, there exists an L-projective set of formulas $\vec{\varphi}$ such that $\Pi \subseteq \vec{\varphi} \subseteq \Pi'$. We first define the set of variables

$$X = \{p \in \mathrm{Var}(\Pi) \mid \neg\neg p \to p \notin \Pi'\}.$$

If $\Pi \cup X \vdash_{\mathrm{L}} \bot$, then by construction $\neg\neg p \to p \in \Pi'$ for some $p \in X$, a contradiction. Hence there exists a classical evaluation e satisfying $\Pi \cup X$. In particular, if $e(p) = 0$, then $\neg\neg p \to p \in \Pi'$. We define the set of formulas

$$\vec{\varphi} = \Pi' \setminus \{\neg\neg p \to p \mid e(p) = 1\}.$$

We also define the substitution

$$\sigma p = \begin{cases} \vec{\varphi} \to p & \text{if } e(p) = 1 \\ (\vec{\varphi} \to p \to \bot) \to \bot & \text{if } e(p) = 0. \end{cases}$$

It remains then to check that σ is an L-projective unifier for $\vec{\varphi}$. For each variable p we need

$$\vec{\varphi} \vdash_{\mathrm{L}} \sigma p \to p \quad \text{and} \quad \vec{\varphi} \vdash_{\mathrm{L}} p \to \sigma p.$$

The only tricky case is to show that $\vec{\varphi} \vdash_{\mathrm{L}} \sigma p \to p$ when $e(p) = 0$. Notice first that $\vec{\varphi} \vdash_{\mathrm{L}} \neg p \to \vec{\varphi} \to \neg p$. Hence also $\vec{\varphi} \vdash_{\mathrm{L}} ((\vec{\varphi} \to p \to \bot) \to \bot) \to \neg\neg p$. But since $e(p) = 0$, $\neg\neg p \to p \in \vec{\varphi}$. So $\vec{\varphi} \vdash_{\mathrm{L}} ((\vec{\varphi} \to p \to \bot) \to \bot) \to p$ as required.

Now we show that $\vdash_{\mathrm{L}} \sigma\varphi$ for each $\varphi \in \vec{\varphi}$, considering all possibilities. First, if φ is of the form $\neg\neg p \to p$, then $e(p) = 0$ and we can use the fact that $\vdash_{\mathrm{L}} \neg\neg\neg\psi \to \neg\psi$ for $\psi = \vec{\varphi} \to p \to \bot$. Otherwise, φ is a simple formula from Π and so $e(\varphi) = 1$. We distinguish two cases:

(1) φ is $\vec{p} \to \bot$. Then $e(p) = 0$ for some $p \in \vec{p}$. Since $\vec{\varphi} \vdash_{\mathrm{L}} \sigma q \to q$ for each $q \in \vec{p}$, we get $\vec{\varphi} \vdash_{\mathrm{L}} \sigma\vec{p} \to p \to \bot$. So by the deduction theorem,

$$\vdash_{\mathrm{L}} \sigma\vec{p} \to \vec{\varphi} \to p \to \bot.$$

Hence also, since $\vdash_{\mathrm{L}} \chi \to \neg\neg\chi$ for any formula χ,

$$\vdash_{\mathrm{L}} \sigma\vec{p} \to ((\vec{\varphi} \to p \to \bot) \to \bot) \to \bot.$$

That is, $\vdash_{\mathrm{L}} \sigma\vec{p} \to \sigma p \to \bot$ as required.

(2) φ is $\vec{\psi} \to p$. If $e(p) = 1$, then $\sigma p = \vec{\varphi} \to p$. Since $\vec{\varphi} \vdash_{\mathrm{L}} \sigma\psi \to \psi$ for each $\psi \in \vec{\psi}$, we obtain $\vec{\varphi} \vdash_{\mathrm{L}} \sigma\vec{\psi} \to p$, and by the deduction theorem, $\vdash_{\mathrm{L}} \sigma\vec{\psi} \to \sigma p$ as required. Otherwise, $e(\psi) = 0$ for some $\psi \in \vec{\psi}$. Again, we distinguish two options:

(i) ψ is q. Then $\vec{\varphi} \vdash_{\mathrm{L}} \vec{\psi} \to q \to p$, so using projectivity and the deduction theorem,

$$\vdash_{\mathrm{L}} \sigma\vec{\psi} \to \vec{\varphi} \to q \to \sigma p.$$

But then also

$$\vdash_{\mathrm{L}} \sigma\vec{\psi} \to ((\vec{\varphi} \to q \to \bot) \to \bot) \to \neg\neg\sigma p.$$

Since $e(p) = 0$, $\vdash_{\mathrm{L}} \neg\neg\sigma p \to \sigma p$, so as required

$$\vdash_{\mathrm{L}} \sigma\vec{\psi} \to ((\vec{\varphi} \to q \to \bot) \to \bot) \to \sigma p.$$

(ii) ψ is $r \to q$. Then $e(r) = 1$ and $e(q) = 0$ and as before

$$\vdash_{\mathrm{L}} \vec{\varphi} \to \sigma\vec{\psi} \to (r \to q) \to \sigma p.$$

But then

$$\vdash_{\mathrm{L}} \sigma\vec{\psi} \to \neg\sigma p \to \vec{\varphi} \to \neg(r \to q).$$

However, it can be checked that

$$\vdash_{\mathrm{IPC}} (\vec{\varphi} \to \neg(r \to q)) \to \neg((\vec{\varphi} \to r) \to ((\vec{\varphi} \to q \to \bot) \to \bot)).$$

Hence, we obtain

$$\vdash_{\mathrm{L}} \sigma\vec{\psi} \to \neg\neg(\sigma r \to \sigma q) \to \neg\neg\sigma p,$$

and since $\vdash_{\mathrm{L}} \neg\neg\sigma p \to \sigma p$, we have $\vdash_{\mathrm{L}} \sigma\vec{\psi} \to (\sigma r \to \sigma q) \to \sigma p$ as required. $\square$

Theorem 3.5. *W is a basis for $\mathrel{|\!\sim}_{\mathrm{L}}$ over* $\mathrm{L_m}$.

Proof. We show that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ iff $\Gamma \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$. One direction was established in Lemma 3.1. To prove the second, suppose that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$. By Lemma 3.2, we can construct a finite set of simple formulas Π such that $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$. But then by Lemma 3.4, $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for all $\Pi' \in \Psi_\Pi$. Hence by Lemma 3.3, $\Pi \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$ and Lemma 3.2 completes the proof that $\Gamma \vdash_{\mathrm{L}_\mathrm{m}^W} \Delta$. $\square$

3.3. A single-conclusion basis

For single-conclusion rules, we define

$$W' = \{(W'_n) \mid n \in \mathbb{N}\}.$$

Theorem 3.6. *W' is a basis for $(\mid\!\sim_{\mathrm{L}})_s$ over* L.

Proof. Suppose that $\Gamma \mid\!\sim_{\mathrm{L}} \varphi$. As in the multiple-conclusion case, we may assume that Γ consists only of simple formulas. Recall the construction above of $X_1, \ldots, X_n$ and the sets of sets of formulas $\Psi_0, \ldots, \Psi_n$. We prove $\Gamma \vdash_{\mathrm{L}W'} \vec{\psi} \to \varphi$ for all $\vec{\psi} \in \Psi_i$ for $i = 0, \ldots, n$ by induction on $n - i$. For the base case, note that by Lemma 3.4, $\Gamma, \vec{\psi} \vdash_{\mathrm{L}} \varphi$ for each $\vec{\psi} \in \Psi_n$ and hence by the deduction theorem, $\Gamma \vdash_{\mathrm{L}} \vec{\psi} \to \varphi$. For the induction step, suppose that $\vec{\psi} \in \Psi_{i-1}$. By the induction hypothesis, for each $p \in X_i$,

$$\Gamma \vdash_{\mathrm{L}W'} (\neg\neg p \to p) \to \vec{\psi} \to \varphi.$$

But by W', substituting $\vec{\psi} \to \varphi$ for q,

$$(X_i \to \bot), \{(\neg\neg p \to p) \to \vec{\psi} \to \varphi \mid p \in X_i\} \vdash_{\mathrm{L}W'} \vec{\psi} \to \varphi.$$

Hence, since $\Gamma \cup X_i \vdash_{\mathrm{L}} \bot$ and by the deduction theorem, $\Gamma \vdash_{\mathrm{L}} X_i \to \bot$,

$$\Gamma \vdash_{\mathrm{L}W'} \vec{\psi} \to \varphi.$$

So, finally, since $\Psi_0 = \{\emptyset\}$, $\Gamma \vdash_{\mathrm{L}W'} \varphi$ as required. $\square$

This result allows us to give the following characterization of the (hereditarily) structurally complete implication–negation fragments of intermediate logics.

Theorem 3.7. L *is (hereditarily) structurally complete if and only if $W' \subseteq$* L.

Proof. If L is (hereditarily) structurally complete, then $W' \subseteq$ L since each rule in W' is L-admissible and therefore also L-derivable. For the other direction, suppose that $W' \subseteq$ L. Then since W' is a basis for the admissible single-conclusion rules of L, this logic and all of its axiomatic extensions are structurally complete. But if all *axiomatic extensions* of a logic are structurally complete, then the logic is hereditarily structurally complete [22, Theorem 2.6]. $\square$

4. Complexity

An analysis of our proofs also solves the complexity problem for the admissible rules of the implication–negation fragment $\mathrm{IPC}{\restriction_{\to,\neg}}$ of intuitionistic logic. We define

$$F(\Gamma) = \{Y \subseteq \mathrm{Var}(\Gamma) \mid (\forall X)(\Gamma, X \vdash_{\mathrm{L}} \bot \Rightarrow X \cap Y \neq \emptyset)\}$$
$$\Psi'_\Gamma = \{\Gamma \cup \{\neg\neg p \to p \mid p \in Y\} \mid Y \in F(\Gamma)\}.$$

Lemma 4.1. *Let Γ be a finite set of simple formulas. Then $\Gamma \mid\!\sim_{\mathrm{L}} \Delta$ iff $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for each $\Pi' \in \Psi'_\Gamma$.*

Proof. Suppose first that $\Gamma \mid\!\sim_{\mathrm{L}} \Delta$. Notice that for each $\Pi' \in \Psi'_\Gamma$ there is a $\Pi \in \Psi_\Gamma$ such that $\Pi' \supseteq \Pi$, so by Lemma 3.4, $\Pi' \vdash_{\mathrm{L_m}} \Delta$ for each $\Pi' \in \Psi'_\Gamma$. Since $\Psi_\Gamma \subseteq \Psi'_\Gamma$, the reverse direction follows by Lemma 3.3 and Theorem 3.5. $\square$

Lemma 4.2. *Let Γ be a finite set of simple formulas. Then deciding $\Pi \in \Psi'_\Gamma$ is solvable in non-deterministic polynomial time (with respect to the size of Γ).*

Proof. We show that $\Pi = \Gamma \cup \{\neg\neg p \to p \mid p \in Y\} \in \Psi'_\Gamma$ iff $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$, which reduces the problem to satisfiability in classical logic. From the construction of Ψ'_n we know that $\Pi \in \Psi'_\Gamma$ iff $Y \in F(\Gamma)$. To complete the proof we show that $Y \in F(\Gamma)$ iff $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$. For the first direction, assume that $Y \in F(\Gamma)$. Since $(\mathrm{Var}(\Gamma) \setminus Y) \cap Y = \emptyset$, we obtain $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$. For the converse direction, assume that $\Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot$ and $\Gamma, X \vdash_{\mathrm{L}} \bot$. Then $X \not\subseteq \mathrm{Var}(\Gamma) \setminus Y$. That is, $X \cap Y \neq \emptyset$. $\square$

Theorem 4.3. *The set of admissible rules of* $\mathrm{IPC}{\restriction_{\to,\neg}}$ *is* PSPACE-*complete.*

Proof. PSPACE-hardness follows from the fact that the set of theorems for this fragment of IPC is already PSPACE-hard [27]. Next, observe that Lemma 3.2 reduces the problem to the problem of checking the $\mathrm{IPC}{\restriction_{\to,\neg}}$-admissibility of rules with a simple set of premises. This reduction is clearly polynomial (by inspection of the proof). To solve this problem we use the contrapositive version of Lemma 4.1. Consider a rule $\Gamma \,/\, \Delta$ with simple premises. First we observe that all $\Pi \in \Psi'_\Gamma$ are of polynomial size with respect to Γ. Thus, to show that $\Gamma \,/\, \Delta$ is *not* $\mathrm{IPC}{\restriction_{\to,\neg}}$-admissible we can non-deterministically guess some $X \subseteq \mathrm{Var}(\Gamma)$ and $\varphi \in \Delta$ and check whether $\Pi = \Gamma, \{\neg\neg p \to p \mid p \in X\} \in \Psi'_\Gamma$ and the IPC-non-derivability of $\Pi \,/\, \varphi$, a problem in PSPACE. Finally, we use the fact that coNPSPACE = PSPACE. $\square$

Clearly, a similar analysis can be used to obtain complexity bounds for the admissible rules of other intermediate logics when bounds are known already for the derivability problem. We remark moreover that although in the case of $\mathrm{IPC}{\restriction_{\to,\neg}}$ the complexity of admissibility matches the complexity of derivability, this is not always the case. Indeed, in contrast to the PSPACE-completeness of derivability, admissibility in full intuitionistic logic is co-NEXP-complete [16].

5. Kripke frames

We refer the reader to [2] for standard definitions and further details regarding Kripke frames and models for intermediate logics. In particular, recall that a frame is Church–Rosser if every finite set of elements with a lower bound also has an upper bound. Weakening this condition, let us call a frame *n-almost-Church–Rosser (n-aCR)* if each set of at most *n non-maximal* elements which has a lower bound also has an upper bound. A frame F is called *almost-Church–Rosser (aCR)* if it is n-aCR for all $n \in \mathbb{N}$.

Lemma 5.1. *(W_n') is valid in a frame F iff F is n-aCR ($n = 2, 3, \ldots$).*

Proof. First suppose that the frame $F = (X, \leq)$ is not n-aCR. Then there exists some set $Y = \{x_1, \ldots, x_m\} \subseteq X$ ($m \leq n$) of non-maximal elements with a lower bound z but no upper bound. We show that (W_m') is not valid in F and thus also (W_n') is not valid in F. For each $i \leq m$ we define $x \models p_i$ iff $x > x_i$. Since Y has no upper bound, $x \not\models \bigwedge_{i=1}^m p_i$ for each $x \in X$, and hence also $z \models \neg \bigwedge_{i=1}^m p_i$. Moreover, since x_i is non-maximal and $x \models p_i$ for all $x > x_i$, we have $x \not\models \neg p_i$ for all $x \geq x_i$ for $i = 1, \ldots, m$. Hence also $x_i \models \neg\neg p_i$ for $i = 1, \ldots, m$. But then $x_i \not\models \neg\neg p_i \to p_i$ and thus also $z \not\models \neg\neg p_i \to p_i$ for $i = 1, \ldots, m$. Finally, we define $x \models q$ iff $x \neq z$ and obtain $z \models (\neg\neg p_i \to p_i) \to q$ for $i = 1, \ldots, m$ which completes the proof.

For the opposite direction, suppose that $F = (X, \leq)$ is n-aCR but (W_n') is not valid in F. Then there is an evaluation $\models$ and a world z such that:

1. $z \not\models q$,
2. $z \models \neg \bigwedge_{i \leq n} p_i$,
3. $z \models (\neg\neg p_i \to p_i) \to q$ for $i = 1, \ldots, n$.

From the last condition we obtain that for each $i = 1, \ldots, n$ there is a world $x_i \geq z$ such that $x_i \models \neg\neg p_i$ and $x_i \not\models p_i$. But this implies the existence of a world $x_i' > x_i$ (since otherwise $x_i \models \neg p_i$ and so $x_i \not\models \neg\neg p_i$). Hence, we have a set of non-maximal elements $\{x_1, \ldots, x_n\}$ with lower bound z. Since F is n-aCR, Y has also an upper bound x. There are two possibilities. First, assume that there is a maximal element $y \geq x$. Then $y \geq x_i$ and $y \models p_i$ (because it is maximal and $x_i \models \neg\neg p_i$) for $i = 1, \ldots, n$. But then we have a contradiction with $y \models \neg \bigwedge_{i \leq n} p_i$. Now assume that there is no maximal element greater than x. We know that for each i there exists $y_i \geq x$ such that $y_i \models p_i$. By assumption, $\{y_1, \ldots, y_n\}$ is a set of non-maximal elements with lower bound z and so, since the frame is n-aCR, this set has an upper bound y. Clearly $y \models p_i$ for each $i = 1, \ldots, n$ and we again have a contradiction. □

Corollary 5.2. *A frame F is aCR iff F validates W'.*

Now recall that an L-*frame* is a frame that validates all the theorems of L and hence, by the deduction theorem, also all the rules of L.

Theorem 5.3. *Let* L *be the implication–negation fragment of some intermediate logic* L′. *Then* L *is (hereditarily) structurally complete iff all* L*-frames are aCR.*

Proof. First observe that we can assume that L′ is axiomatized over IPC by formulas involving implication and negation only, and thus by McKay's theorem [18], L′ is Kripke complete and so is L. By Theorem 3.7, L is (hereditarily) structurally complete iff $W' \subseteq$ L. But since L is Kripke complete, $W' \subseteq$ L iff all L-frames validate W' iff, by Corollary 5.2, all L-frames are aCR. □

Our results allow us to determine (hereditary) structural completeness for the implication–negation fragments of many well-studied intermediate logics. In particular, this fragment is hereditarily structurally complete for any Gödel logic (as was already well-known; see, e.g., [4]). Also, De Morgan (Jankov) logic which like IPC has admissible underivable rules in the full logic (and indeed shares a basis; see [10]) is hereditarily structurally complete for this fragment. On the other hand, Gabbay–de Jongh logics (complete with respect to the class of finite trees in which every point has at most $n + 1$ immediate successors) and the logics of frames of depth at most n are not structurally complete in this fragment for $n \geq 2$. Moreover, we can prove the following general result:

Theorem 5.4. *The implication–negation fragment of any intermediate logic with the disjunction property is not structurally complete.*

Proof. Suppose that L is the implication–negation fragment of an intermediate logic L′ with the disjunction property. If L is structurally complete, then in particular (W_2') is L-derivable. That is, $(p_1 \to p_2 \to \bot), ((\neg\neg p_1 \to p_1) \to q), ((\neg\neg p_2 \to p_2) \to q) \vdash_{\mathrm{L}} q$. Hence, substituting $(\neg\neg p_1 \to p_1) \vee (\neg\neg p_2 \to p_2)$ for q and applying the deduction theorem,

$$\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to ((\neg\neg p_1 \to p_1) \vee (\neg\neg p_2 \to p_2)).$$

However, the independence of premises rule is admissible for any intermediate logic with the disjunction property [20]. Hence $\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to (\neg\neg p_1 \to p_1)$ or $\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to (\neg\neg p_2 \to p_2)$. So by substituting $\bot$ for p_2 or p_1, $\vdash_{\mathrm{L}'} \neg\neg p \to p$. That is, L′ is classical logic, contradicting the disjunction property. □

In particular, Medvedev logic (the logic of frames consisting of non-empty subsets of $\{1, \dots, n\}$ dually ordered by inclusion) is not structurally complete, unlike the logic in the full language which is structurally complete but not hereditarily structurally complete [24].

Note finally that certain logics may require only a finite basis. In particular, for the implication–negation fragments of the logics of frames with at most n nodes, the single rule (W_n) suffices. This raises then the question as to whether the admissible rules of the implication–negation fragment of intuitionistic logic might also have a finite basis: the answer is no.

Theorem 5.5. *The set of single-conclusion admissible rules of* $\mathrm{IPC}{\restriction_{\to,\neg}}$ *has no finite basis.*

Proof. It is enough to show that for each $n \geq 2$, the rules $\{(W_i') \mid 0 \leq i \leq n\}$ do not form a basis (since any finite set of rules provable in $\mathrm{IPC} + W'$ would be provable in $\mathrm{IPC} + \{(W_i') \mid 0 \leq i \leq n\}$ for some n). From McKay's theorem we know that the logic $\mathrm{L} = \mathrm{IPC} + \{(W_i') \mid 0 \leq i \leq n\}$ is Kripke complete with respect to the class of all n-aCR Kripke frames. Moreover, there exists an n-aCR frame which is not $n + 1$-aCR. Hence (W_{n+1}') is not derivable in L. $\square$

6. Unification type

Let us briefly recall some standard definitions regarding the unification type of a logic, noting that for an algebraizable logic such as L (a consistent axiomatic extension of the implication–negation fragment of IPC), these coincide with the definitions for the corresponding class of algebras (see [1] for further details). Let Γ be a finite set of implication–negation formulas and σ_1, σ_2 two L-unifiers for Γ. We define

$$\sigma_1 \leq_{\mathrm{L}} \sigma_2 \quad \text{iff} \quad \text{there is a substitution } \sigma \text{ such that } \sigma_2(p) = \sigma(\sigma_1(p)) \quad \text{for all } p \in \mathrm{Var}(\Gamma).$$

A *complete set of* L*-unifiers* for Γ is a set $\mathcal{C}$ of L-unifiers for Γ such that for any L-unifier σ for Γ, there exists $\sigma' \in \mathcal{C}$ such that $\sigma' \leq_{\mathrm{L}} \sigma$. $\mathcal{C}$ is called *minimal* if additionally, for any $\sigma_1, \sigma_2 \in \mathcal{C}$, if $\sigma_1 \leq_{\mathrm{L}} \sigma_2$, then $\sigma_1 = \sigma_2$. A substitution σ is called a *most general* L*-unifier* of Γ iff $\{\sigma\}$ is a (minimal) complete set of L-unifiers for Γ.

Γ is said to have *unitary* (*finitary*, *infinitary*) type iff it has a minimal complete set of L-unifiers of cardinality 1 (finite cardinality, infinite cardinality), and type *zero* if it does not have a minimal complete set of L-unifiers. The unification type of L is the maximal type of a finite set of implication–negation formulas according to the ordering *unitary* < *finitary* < *infinitary* < *zero*.

Theorem 6.1. *If* L *is strictly contained in classical logic, then* L *has finitary unification type.*

Proof. We show first that L has unitary unification type if and only if it is classical logic (announced by Wroński in [29]). The fact that classical logic has unitary unification type is well-known (see, e.g., [1]). We prove the converse. Consider the formula $p \to (q \to \bot)$ and L-unifiers for this formula σ_1, σ_2 defined by $\sigma_1(p) = \bot$, $\sigma_1(q) = q$, $\sigma_2(p) = p$, $\sigma_2(q) = \bot$. Suppose that L has unitary unification type. So there exists a most general unifier σ for $p \to (q \to \bot)$. But then, since $\vdash_{\mathrm{L}} \sigma p \to (\sigma q \to \bot)$, by the admissibility of the rule (W_2), also $\vdash_{\mathrm{L}} \neg\neg\sigma p \to \sigma p$ or $\vdash_{\mathrm{L}} \neg\neg\sigma q \to \sigma q$. Suppose without loss of generality that $\vdash_{\mathrm{L}} \neg\neg\sigma p \to \sigma p$. Since σ is a most general unifier, $\sigma_2 = \sigma'\sigma$ for some substitution σ'. But then $\vdash_{\mathrm{L}} \neg\neg\sigma_2(p) \to \sigma_2(p)$; i.e. $\vdash_{\mathrm{L}} \neg\neg p \to p$. So L is classical logic.

To see that L has finitary unification type, let Γ be a finite set of implication–negation formulas. It suffices to find any finite complete set of L-unifiers for Γ. We obtain first a finite set of simple formulas Π such that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ iff $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \Delta$ (Lemma 3.2). It follows that $\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \varphi$ for all $\varphi \in \Pi$ and $\Pi \mathrel{|\!\sim}_{\mathrm{L}} \psi$ for all $\psi \in \Gamma$. Hence a substitution σ is an L-unifier for Γ iff it is an L-unifier for Π. Recall the finite set Ψ_Π constructed in Section 3.2 and observe that it enjoys the following two properties (the first follows by construction and the second from the proof of Lemma 3.4):

(1) Any L-unifier for Π is an L-unifier for some $\Pi' \in \Psi_\Pi$.
(2) For each $\Pi' \in \Psi_\Pi$, there exists an L-projective set $\vec{\varphi}$ such that $\Pi \subseteq \vec{\varphi} \subseteq \Pi'$.

Let $\mathcal{C}$ be the finite set of substitutions consisting of an L-projective unifier (and hence also a most general unifier) for each $\vec{\varphi}$ identified in (2). We show that $\mathcal{C}$ is a complete set of L-unifiers for Γ. First note that each $\sigma \in \mathcal{C}$ is an L-unifier for Π and hence also for Γ. Now let σ be an L-unifier for Γ. Then σ is also an L-unifier for Π and therefore, by (1), for some $\Pi' \in \Psi_\Pi$. So σ is an L-unifier for the $\vec{\varphi}$ identified for Π' in (2) and hence there exists $\sigma' \in \mathcal{C}$ such that $\sigma' \leq_{\mathrm{L}} \sigma$ as required. $\square$

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