Logic of Programs | **Lecture 21**
CS 6860 Fall 2015 | **Thurs. Nov 5, 2015**

Lecture 21

We will compare the computational content of constructive proofs in iFOL (also called $\mathscr{L}_1$) and intuitionistic (impredicative) second order logic (also called $\mathscr{L}_2$).

We examined the evidence terms for $A \,\&\, B \Rightarrow A \vee B$ in iFOL. We can have essentially these two evidence terms:

$$\lambda(ab.\text{spread}(ab; a, b.\text{inl}(a))) \qquad \text{or} \qquad \lambda(ab.\text{spread}(ab; a, b.\text{inr}(b))).$$

What is the "second-order realizer"? Is it "polymorphic" or "uniform" as the first-order ones are? *Can we reconstruct the proof from the realizer?*

	$\vdash \forall X\!:\!\text{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X) \Rightarrow \forall X\!:\!\text{Prop}.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$	$\lambda(h.\_\_\_)$ $\uparrow$
$h\!:\!\forall X\!:\!\text{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X)$	$\vdash \forall X\!:\!\text{Prop}.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$	by $\lambda(X.\_\_\_)$
$h\!:\!(...)$, $X\!:\!\text{Prop}$	$\vdash ((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$	$\text{ap}(h; X)$
$X\!:\!\text{Prop}$, $h(X)\!:\!A \Rightarrow (B \Rightarrow X) \Rightarrow X$	$\vdash (A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)$	$\lambda(ax; \_\_\_)$
$X\!:\!\text{Prop}$, $h(X)\!:\!(...)$, $ax\!:\!(A \Rightarrow X)$	$\vdash (B \Rightarrow X) \Rightarrow X$	$\lambda(bx.\_\_\_)$
$X\!:\!\text{Prop}$, $h(X)\!:\!(...)$, $ax\!:\!(A \Rightarrow X)$, $bx\!:\!(B \Rightarrow X)$	$\vdash X$	$\text{ap}(h(X); \_\_\_) = z$
$X\!:\!\text{Prop}$, $ax\!:\!(A \Rightarrow X)$, $bx\!:\!(B \Rightarrow X)$	$\vdash A \Rightarrow (B \Rightarrow X)$	$\lambda(a.\_\_\_)$ $\uparrow$
$X\!:\!\text{Prop}$, $ax\!:\!(A \Rightarrow X)$, $bx\!:\!(B \Rightarrow X)$, $a\!:\!A$,	$\vdash (B \Rightarrow X)$	by bx $\uparrow$
$z\!:\!X$	$\vdash X$	by z

The realizer is $\lambda(h.\lambda(X.\lambda(ax.\lambda(bx.\text{apseq}(h(X); \lambda(a.bx); z.z)))))$

What disjunct did we prove?

How could we prove the other one?

We can prove either given h. If we want A, we use $h(A)$, which is:

$$\lambda(ax.\lambda(bx.\text{apseq}(h(A); \lambda(a.bx); z.z))).$$

We need $ax \in A \Rightarrow A$ and $bx \in (B \Rightarrow A)$ which must be $\lambda(x.a)$.

If we want B, we use $h(B)$ and we need $A \Rightarrow B$ and $B \Rightarrow B$. So we need $\lambda(x.b)$ for $b \in B$. We can't have empty A or B since we need $A \Rightarrow (B \Rightarrow B) \Rightarrow B$ or $A \Rightarrow (B \Rightarrow A) \Rightarrow A$.

We can prove the following illuminating fact in Nuprl.

Theorem. $\forall A, B\!:\!\text{Prop}_1.(A \vee B) \Leftrightarrow \forall X\!:\!\text{Prop}_1.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$

Proof.

The direction $(A \vee B) \Rightarrow \forall X\!:\!\text{Prop}_1.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$ is trivial.

We have the hypothesis $d\!:\!A \vee B,\ X\!:\!\text{Prop},\ ax\!:\!A \Rightarrow X,\ bx\!:\!B \Rightarrow X \vdash X$.

We use the case split on d, decide($d; a.\_\_\_; b.\_\_\_$). In case $a\!:\!A$ we use $A \Rightarrow X$ and in case b, $B \Rightarrow X$.

The direction $\Leftarrow$ is more subtle.

We have the hypotheses:

$A : \text{Prop}_1,\ B : \text{Prop}_1,\ h : \forall X : \text{Prop}_1.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X))$ to show $A \vee B$. We substitute $(A \vee B)$ for X in h to get the additional hypothesis.

$$h'\!:\!A \Rightarrow (A \vee B) \Rightarrow (B \Rightarrow (A \vee B)) \Rightarrow X \vdash X.$$

Then we show $A \Rightarrow (A \vee B)$ to get the hypothesis $(B \Rightarrow (A \vee B)) \Rightarrow X$ and finally show $B \Rightarrow (A \vee B)$ to conclude X.

QED.