Logic of Programs	**Lecture 20**
CS 6860 Fall 2015	**Tues. Nov 3, 2015**

Lecture 20

Second-order Intuitionistic Logic, $\mathscr{L}_2$, continued.

It is interesting to compare the first and second-order intuitionistic logics. The second order logic is extremely parsimonious. The standard version as in Prawitz, system F, and the theory of species are *impredicative*. We can also define a stratified (predicative) version inside Nuprl. We have already seen that $\mathscr{L}_2$ allows a very elegant and simple treatment of False, as $\forall X\!:\!\mathrm{Prop}.X$. This account does not rely on exceptions. The account of disjunction does not tempt us to think about uniformity. This raises the question of whether we can find another source of it in $\mathscr{L}_2$ and whether a uniform universal quantifier is conceptually interesting in $\mathscr{L}_2$. It suggests a less potent use of impredicativity. We might call such a logic *uniform* $\mathscr{L}_2$. Is this an interesting notion? Could it enrich the Coq impredicative type Prop, which is used to provide classical treatment of logic?

The theory $\mathscr{L}_2$ is pedagogically interesting for learning about the axiomatization of iFOL and its constructive semantics in constructive type theory. To develop a sense of this we contrast proofs of simple logical facts in iFOL vs $\mathscr{L}_2$. We also contrast the realizers and the *evidence semantics*.

Recall:

$$\begin{array}{rcl} A \,\&\, B & == & \forall X\!:\!\mathrm{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X) \\ A \vee B & == & \forall X\!:\!\mathrm{Prop}.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)) \\ \exists x\!:\!D.A & == & \forall X\!:\!\mathrm{Prop}.(A(x) \Rightarrow X) \Rightarrow X) \end{array}$$

Here are some sample $\mathscr{L}_2$ proofs constructed with the iFOL proofs.

1.

$$A \,\&\, B \Rightarrow A \qquad \lambda(ab.\mathrm{spread}(ab.a, b.a))$$

versus

$$\begin{array}{rll} \vdash \forall X\!:\!\mathrm{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X) \Rightarrow A & & \lambda(h.__) \\ h\!:\!\forall X\!:\!\mathrm{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X) & \vdash A & \mathrm{ap}(h; A) \\ h'\!:\!A \Rightarrow (B \Rightarrow A) \Rightarrow A & \vdash A & \mathrm{ap}(h'; __) = v \\ & \vdash A \Rightarrow (B \Rightarrow A) & \lambda(x.__) \\ x\!:\!A & \vdash B \Rightarrow A & (y.__) \\ x\!:\!A, y\!:\!B & \vdash A & \text{by } x\,\hat{\uparrow} \\ v\!:\!A & \vdash A & \text{by } v \end{array}$$

2.

$$A \Rightarrow (A \vee B) \qquad \lambda(a.\mathrm{inl}(a))$$

versus

$$\begin{array}{rll}
\vdash A \Rightarrow \forall X{:}\mathrm{Prop}.((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X) & & \lambda(x.__) \\
x{:}A & \vdash \forall X{:}\mathrm{Prop}.((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X) & \lambda(X.__) \\
x{:}A,\ X{:}\mathrm{Prop} & \vdash ((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X) & \lambda(f.\lambda(g.__)) \\
x{:}A,\ X{:}\mathrm{Prop},\ f{:}(A \Rightarrow X),\ g{:}(B \Rightarrow X) & \vdash X & \mathrm{ap}(f;__) \\
 & \vdash A & \text{by } x \\
v{:}X & \vdash X & \text{by } v
\end{array}$$

We can express this in the predicative logic of Nuprl using Prop_i (also denoted $\mathbb{U}_i$). We can also express this using the intersection over a family, e.g.

$$A \Rightarrow \bigcap_{X:\mathbb{U}_i} (A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)$$

3.

$$A \& B \Rightarrow A \vee B \quad \lambda(ab.\mathrm{spread}(ab; a, b.\mathrm{inl}(a)))$$

or

$$\lambda(ab.\mathrm{spread}(ab; a, b.\mathrm{inr}(b)))$$

versus

$$\begin{array}{rll}
\vdash \forall X{:}\mathrm{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X) \Rightarrow \forall X{:}\mathrm{Prop}.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)) & & \lambda(h.__) \\
h{:}\forall X{:}\mathrm{Prop}.(A \Rightarrow (B \Rightarrow X) \Rightarrow X) & \vdash \forall X{:}\mathrm{Prop}.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)) & \text{by } \lambda(X.__) \\
h{:}(...),\ X{:}\mathrm{Prop} & \vdash ((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)) & \mathrm{ap}(h; x) \\
X{:}\mathrm{Prop},\ h(X){:}A \Rightarrow (B \Rightarrow X) \Rightarrow X & \vdash (A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X) & \lambda(ax;__) \\
X{:}\mathrm{Prop},\ h(X){:}(...),\ ax{:}(A \Rightarrow X) & \vdash (B \Rightarrow X) \Rightarrow X & \lambda(bx.__) \\
X{:}\mathrm{Prop},\ h(X){:}(...),\ bx{:}(B \Rightarrow X) & \vdash X & \mathrm{ap}(h(X);__) = z \\
X{:}\mathrm{Prop},\ bx{:}(B \Rightarrow X) & \vdash A \Rightarrow (B \Rightarrow X) & \lambda(a.__) \\
a{:}A,\ bx{:}(B \Rightarrow X) & \vdash (B \Rightarrow X) & \text{by } bx \\
z{:}X & \vdash X &
\end{array}$$