Lecture 19

We will examine second order intuitionistic logic more closely.[1] The system we examine is close to the logic called *system F* studied extensively by Girard [1]. Troelstra [4] discusses second order intuitionistic logic and cites Prawitz [2]. The only second order logical operator we need is second order universal quantification, $\forall X : \text{Prop}.F(X)$.[2]

Prawitz defines the *second-order* existential quantifier as

$$\exists X : \text{Prop}.A(X) == \forall X : \text{Prop}.(\forall Y : \text{Prop}.(A(Y) \Rightarrow X) \Rightarrow X).$$

To continue our definition of the first-order logical operators, we look at how to derive the rules for the *first-order* existential quantifier defined as

$$\exists x : D.A(x) == \forall X : \text{Prop}.(\forall x : D.(A(x) \Rightarrow X) \Rightarrow X).$$

We expect to be able to derive this introduction rule.

$H \vdash \exists x : D.A(x)$
$H \vdash D$ by d
$H \vdash A(d)$ by a

Here is the second order version:

$H \vdash \forall X : \text{Prop}.\forall x : D.(A(x) \Rightarrow X) \Rightarrow X$

$H,\ X : \text{Prop},\ \text{all}: \forall x : D.(A(x) \Rightarrow X)$	$\vdash X$	by ap(all;___)
	$\vdash D$	by d - - - - - ↑
$d : D,\ f : A(d) \Rightarrow X$	$\vdash X$	by ap(f; ___)
	$\vdash A(d)$	by a - - - - - ↑
$v : X$	$\vdash X$	

We look again more closely at the case of "or", $A \vee B$. Next week we will show how to express these results uniformly.

[1]There is a related theory in Stenlund [3] called the Theory of Species. That theory also includes lambda terms with first and second order variables, e.g. $\lambda(x.b(x))$, $\lambda(X.b(X))$.

[2]Bertrand Russell knew the definition of False and of, &, in his Principles of Math, 1903.

Intuitionistic Second-Order Logic

We have already shown how to prove the following:

1. $A \Rightarrow (A \vee B)$

$$\begin{array}{rll} & \vdash A \Rightarrow \forall X\!:\!\mathrm{Prop}.((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)) & \lambda(a.\lambda(X.\_\_\_\_\_)) \\ a\!:\!A,\ X\!:\!\mathrm{Prop} & \vdash (A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X) & \lambda(f.\_\_\_) \\ f\!:\!A \Rightarrow X & \vdash (B \Rightarrow X) \Rightarrow X & \lambda(g.\_\_\_) \\ g\!:\!(B \Rightarrow X) & \vdash X & \text{by } \mathrm{ap}(f;a) = v \\ v\!:\!X & \vdash X & \end{array}$$

The evidence term is $\lambda(a.\lambda(X.\lambda(f.\lambda(g.\mathrm{ap}(f;a)))))$

In iFOL the evidence is simply $\lambda(x.\mathrm{inl}(x))$.

2. Here is the elimination rule.

$((A \Rightarrow C) \,\&\, (B \Rightarrow C)) \Rightarrow (A \vee B) \Rightarrow C$

$$\begin{array}{rll} ac\!:\!(A \Rightarrow C),\ bc\!:\!(B \Rightarrow C) & \vdash \forall X\!:\!\mathrm{Prop}(\_\_\_) \Rightarrow C & \lambda(\mathrm{or}.\_\_\_) \\ \mathrm{or}\!:\ \forall X\!:\!\mathrm{Prop}.(A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X) & \vdash C & \mathrm{ap}(\mathrm{or};C) \\ \mathrm{orap} : (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C & \vdash C & \mathrm{ap}(\mathrm{orap};\_\_\_) \\ & \vdash (A \Rightarrow C) & \text{by } ac \\ v\!:\!(B \Rightarrow C) \Rightarrow C) & \vdash C & \mathrm{ap}(v, bc) = z \\ & \vdash (B \Rightarrow C) & \text{by } bc \\ z\!:\!C & \vdash C & \text{by } z \end{array}$$

References

[1] J-Y. Girard, P. Taylor, and Y. Lafont. *Proofs and Types*, volume 7 of *Cambridge Tracts in Computer Science*. Cambridge University Press, 1989.

[2] D. Prawitz. *Natural Deduction*. Dover Publications, New York, 1965.

[3] S. Stenlund. *Combinators, λ–Terms, and Proof Theory*. D. Reidel, Dordrechte, 1972.

[4] Anne Sjerp Troelstra. *Metamathematical Investigation of Intuitionistic Mathematics*, volume 344 of *Lecture Notes in Mathematics*. Springer-Verlag, 1973.