Logic of Programs	Lecture 18
CS 6860 Fall 2015	Tues. Oct 26, 2015

Lecture 18

Ideas about Intuitionism from Michael Dummett's *Elements of Intuitionism*, Oxford University Press, 1977. All page numbers refer to this book.

"Nowhere in the whole field of mathematical logic and of foundations of mathematics are such deep philosophical issues involved as in the study of intuitionism.", p.ix.

"...intuitionism will never succeed in its struggle against rival, and more widely accepted, forms of mathematics unless it can win the philosophical battle." p. viii.

Dana Scott encouraged him to write this when Dana was a professor at Oxford.

In the early part of the 20th century, there was a great deal of philosophical work on the foundations of mathematics. The work of Frege, Russell, and Wittgenstein are exmples. Dummett says on page 1, "only the intuitionist system originated by Brouwer survives today as a viable theory..."

The other serious contenders were the Frege/Russell attempt to reduce mathematics to logic and Hilbert's program of basing mathematics on formal systems, which could be shown to be consistent using only "finitistic" reasoning – a form of logical reasoning that is more limited than intuitionistic reasoning. This topic remains a technical area of logic, proof theory, but it does not provide a philosophical grounding for mathematics as Hilbert has imagined, because Gödel incompleteness theorems demonstrated that Hilbert's program could not work.

Dummett stresses a point by Kreisel that intuitionistic philosophy comprises two theses (see p. 360).

Positive:	Intuitionistic meaning is coherent and legitimate.
Negative:	The classical philosophy is incoherent. I believe that *virtual evidence semantics* shows this to be unsupportable. Kreisel also takes this view for other reasons, p. 360.[1] *p.361 Dummett makes a blunder saying, if classical math is also intelligible, then intuitionistic philosophy "loses much of its point."

In order to resolve the completeness issue for iFOL, we need to be clearer about the metatheory and the precise semantics of first-order logic. One of the earliest attempts to give a precise semantics close to the basic intuitions was based on the idea of defining the logical operators, &, $\vee$, $\exists$, and False in terms of *constructive implication* and *universal quantification* because those operators can be explained in terms of computable functions. This approach also fo-

[1] We willl examine this new approach later in the course. It uses a key idea from constructive type theory.

cuses attention on the critical role of Church's Thesis that we have seen is fundamental to the completeness question.

The semantics of &, $\vee$, $\exists$, and False can be made both constructive and abstract by defining them in *intuitionistic second order logic*, $\mathscr{L}_2$, as suggested by Prawitz. Here are his definitions:

$$\begin{array}{rcl}
A\&B & == & \forall X\!:\!\text{Prop.}(A \Rightarrow (B \Rightarrow X) \Rightarrow X) \\
A \vee B & == & \forall X\!:\!\text{Prop.}(A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X) \\
\exists x.A & == & \forall X\!:\!\text{Prop.}\forall x.(A(x) \Rightarrow X) \Rightarrow X \\
\text{False} & == & \forall X\!:\!\text{Prop.}\ X
\end{array}$$