Logic of Programs	**Lecture 16**
CS 6860 Fall 2015	**Tues. Oct 20, 2015**

Lecture 16

Intuitionistic First-Order Logic – *i*FOL

We will now advance to first-order logic. We will study the *pure* first-order logic with no constants, no equality, and no function symbols. We are able to prove completeness of this logic, *i*FOL, with respect to uniform evidence. This might seem unexpected in light of the results we cited in Lecture 15 that show *i*FOL to be *incomplete* with respect to the intuitionistic version of classical Tarski semantics. We briefly touched on this semantics in Lecture 14, citing Troelstra and van Dalen for the result that *i*FOL is incomplete for this "standard intuitionistic semantics."

We will look briefly at the incompleteness result since that has received a good deal of attention as a contrast to the Gödel completeness result for classical FOL. The folklore has it that Gödel's result cannot be constructive.[1]

We will not explore *Kripke models* and the important result of Vim Veldman that *i*FOL is complete with respect to "exploding" Kripke models. His proof is constructive. As yet it is not clear how to use his models in a computational way, unlike the situation for uniform evidence semantics.

Completeness

Leivant result 1976

1. Principle of constructivity called RED

   RED: every constructively decidable predicate R over $\mathbb{N}$ is weakly r.e.

2. RED implies that *i*FOL valid formulas are not r.e. assuming Church's thesis.

   He denotes the *intuitionistic theory of species* as $\mathscr{L}_2$. HA is interpretable in $\mathscr{L}_2$.

   $\mathscr{L}_1$ is *i*FOL.

   $\mathscr{L}_2$ is in Prawitz (Ideas and Results 70) p.271.

RED is $\forall x.(P(x) \vee \sim P(x)) \Rightarrow \sim\sim \exists e.\forall n(P(n) \Leftrightarrow \exists b.T(e,n,b))$

We have a function $d\colon D \to P(x) \vee (P(x) \to \bot)$ in any model.

We need to know that there is evidence for $(\exists e.\forall n(P(n) \Leftrightarrow \exists b.T(e,n,b)) \Rightarrow \bot) \Rightarrow \bot$.

[1] Gödel's dissertation ends with remarks: "Essential use is made of the principle of the excluded middle for infinite collections." It is widely believed that it is not possible to prove this theorem constructively.

We would need to compute the index e from the evidence d, but this claim is weaker, we need to show that if we assume $\exists e.\forall n.P(n) \Leftrightarrow (\exists b.T(e,n,b) \Rightarrow \bot)$, then we can find evidence for $\bot$.