Logic of Programs	**Lecture 14**
CS 6860 Fall 2015	**Thurs. Oct 8, 2015**

Lecture 14

We will briefly discuss admissible rules, looking for the right research questions. Our current idea is to try to use these rules to imporve the computational behavior of extracted realizers from admissible rules.

The techniques arising from uniform evidence and the completeness of mPC with respect to uniform evidence have informed our thinking about the computational meaning of mPC and iPC. Therefore, we will start exploring this completeness theorem and the new technical concepts developed for it. This will allow us to explore new ideas that arise from Mark Bickford's efforts to formalize the completeness theorem in Nuprl. We will start to explore his formalization after **fall break** next week.

Our account of completeness begins with the attached lecture notes on the published article *Intuitionistic completeness of first-order logic*.

Intuitionistic Validty from A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction*, Vol. 1, North-Holland, Amsterdam, 1988.

Chapter 2, p. 75:

Given a first-order language $\mathcal{L}$ with relation symbols $\mathrm{R}_1, ..., \mathrm{R}_n$ where R_i has $n(i)$ arguments, and let $\mathrm{A}(\mathrm{R}_1, ..., \mathrm{R}_n)$ be a sentence in $\mathcal{L}$. An *intuitionistic structure (model)* for $\mathcal{L}$ is an $n+1$ tuple $< M, R_1, ..., R_n >$ where M is an intuitionistically meaningful domain, and R_i is an $n(i)$ place relation over M, i.e. $\mathrm{R}_i \subset \mathrm{M}^{n(i)}$. We write $\mathcal{M} =< M, R_1, ..., R_n >$ for the model. $\mathrm{A}(\mathrm{R}_1, ..., \mathrm{R}_n)$ is *intuitionistically valid* in $\mathcal{M}$, $\mathcal{M} \models \mathrm{A}$, iff $A^{\mathrm{M}}(R_1, ..., R_n)$ holds intuitionistically where $\mathrm{A}^{\mathrm{M}}(\mathrm{R}_1, ..., \mathrm{R}_n)$ is obtained from A by replacing all occurences of R_i by R_i and relativizing all quantifiers in A to domain M.

This is the exact same way that classical structures are defined with the logical operators treated intuitionistically.

If $M \models \mathrm{A}$ holds for all intuitionistic structures M, we write $\models \mathrm{A}$.

Vol II p.164. Here is an exact quote from Treolstra and van Dalen:

3.1 The results of the preceding section might lead us to believe that completeness for full IQC (we say iFOL) for the (above) notion of intuitionistic validity is within reach. We shall show that, nevertheless, *we cannot expect to achieve this.* As in the preceding section we concentrate on *pure first-order* predicate logic without equality and function symbols (or constants – where $n(i) = 0$).

Theorem 3.6: Completeness of iFOL implies that for each primitive recursive $A(n, \alpha)$ a *choice sequence* over $\{0, 1\}$ $\forall\alpha \sim\sim \exists n.A(n, \alpha) \Rightarrow \forall\alpha\exists n.A(n, \alpha)$.

In particular, completeness implies Markov's Principle for primitive recursive $A(n, \alpha)$. This principle is denoted MP_{PR}.

Discussion 3.8 (p.699 Vol III). If we assume Church's Thesis for numerical functions, then iFOL is incomplete, see Kreisel 1970, "Church's Thesis is a kind of reducible axiom of constructive mathematics." *Intuitionism and Proof Theory*, 1970.