Logic of Programs	**Lecture 11**
CS 6860 Fall 2015	**Tues. Sept 29, 2015**

Lecture 11

1. **Gilvenko's Theorem** If $\vdash_{PC} A$ then $\vdash_{iPC} \sim\sim A$.

   One idea for the proof is to note that if $P_1, ..., P_n$ are the propositional variables of A, then we know

   $$\vdash_{iPC} (P_1 \vee \sim P_1) \,\&\, ... \,\&\, (P_n \vee \sim P_n) \Rightarrow A$$

   This means we know that $\sim A \Rightarrow \sim ((P_1 \vee \sim P_1), \,\&\, ... \,\&\, (P_n \vee \sim P_n))$.

   Thus we know $\sim\sim ((P_1 \vee \sim P_1) \,\&\, \sim\sim (P_2 \vee \sim P_2) \,\&\, ... \,\&\, \sim\sim (P_n \vee \sim P_n)) \Rightarrow \sim\sim A$

   We also know $\sim\sim (P_i \vee \sim P_i)$ for all P_i.

   Hence $\sim\sim (P_1 \vee \sim P_1) \,\&\, ... \,\&\, \sim\sim (P_n \vee \sim P_n)$.

   But we can show:

   $$\sim\sim (P_1 \vee \sim P_1) \,\&\, ... \,\&\, \sim\sim (P_n \vee \sim P_n) \Rightarrow \sim\sim ((P_1 \vee \sim P_1) \,\&\, ... \,\&\, (P_n \vee \sim P_n))$$

   This is exactly all we need to prove $\sim\sim A$.

2. Recall proof of $\sim\sim (P \vee \sim P)$

   $$\vdash ((P \vee \sim P) \Rightarrow \bot) \Rightarrow \bot \quad \lambda(f.__)$$

$f : (P \vee \sim P) \Rightarrow \bot$	$\vdash \bot$	$\mathrm{ap}(f; __)$	$v = \mathrm{ap}(f; __)$
	$\vdash P \vee \sim P$	$\mathrm{inr}(__)$	
	$\vdash P \Rightarrow \bot$	$\lambda(p.__)$	
$p : P$	$\vdash \bot$	$\mathrm{ap}(f; __)$	$w = \mathrm{ap}(__)$
	$\vdash P \vee \sim P$	by $\mathrm{inl}(p)$	
$w : \bot$	$\vdash \bot$		
$v : \bot$	$\vdash \bot$	hyp v	

   The evidence term (extract of proof) is:

   $$\lambda(f.\mathrm{ap}(f; \mathrm{inr}(\lambda(p.\mathrm{ap}(f; \mathrm{inl}(p))))))$$

   Can we build the proof from the evidence term? **Key question**.

3. Building proofs from evidence – simple example

   - $\sim A \vee \sim B \Rightarrow \sim (A \,\&\, B)$

     $\lambda(d.\lambda(ab.\mathrm{spread}(ab;\, a, b.\mathrm{decide}(d; na.\mathrm{ap}(na; a);\, nb.\mathrm{ap}(nb; b)))))$

     Exercise (done in lecture): Build the proof from the evidence term.

4. There is a curious feature of iPC and other nonclassical theories. It goes under the heading of *admissible rules*. Consider this example for iPC from Mints.

$$\vdash (A \Rightarrow B) \Rightarrow A \vee C \text{ implies } \vdash (A \Rightarrow B) \Rightarrow A \vee (A \Rightarrow B) \Rightarrow C$$

But, we do not have

$$\vdash_{iPC} ((A \Rightarrow B) \Rightarrow A \vee C) \Rightarrow ((A \Rightarrow B) \Rightarrow A) \vee ((A \Rightarrow B) \Rightarrow C)$$

This phenomenon is also visible for

$$(B \Rightarrow (A \vee C)) \Rightarrow (B \Rightarrow A) \vee (B \Rightarrow C)$$

Try a proof:

$$\begin{array}{rl} & \vdash (B \Rightarrow (A \vee C)) \Rightarrow ((B \Rightarrow A) \vee (B \Rightarrow C)) \\ f : B \Rightarrow (A \vee C) & \vdash (B \Rightarrow A) \vee (B \Rightarrow C) \text{ we are stuck!} \end{array}$$

What if we used a new constructor, a parallel **or**.

$$f : B \Rightarrow (A \vee C) \vdash (B \Rightarrow A) \textbf{ or } (B \Rightarrow C)$$

$$\begin{array}{rlll} f : B \Rightarrow (A \vee C), b : B & \vdash A \textbf{ or } C & \text{ap}(f; ___) & v \\ & \vdash B & \text{by } b & \\ v : A \vee C & \vdash A \textbf{ or } C & \text{by decide}(v; a.___, c.___) & \\ a : A & \vdash A \textbf{ or } C & \text{by } a & \\ c : C & \vdash A \textbf{ or } C & \text{by } c & \end{array}$$

We could introduce a parallel decide, $\text{pdecide}(v; a.___; c.___)$.

What is the right approach? Interesting *research problem*.

5. Back to the basics – Brouwer's insights.

   There are three excellent articles on intuitionism in the *Stanford Encyclopedia of Philosophy* (SEP, `http://plato.stanford.edu/index.html`):

   - *Intuitionistic logic development*, Mark van Atten, 2009.
   - *Intuitionistic logic*, Joan Moschovakis, 2015.
   - *Intuitionism in the philosophy of mathematcs*, Rosalie Lemhoff, 2013.

6. Philosophical issues and historical remarks.

   There has been a long and strong tradition of constructive mathematics – starting with the Greeks and gaining depth over the centuries. We associate Kronecker, Poincaré, Henrich, Bishop, Nelson, Edwards, and many others with this tradition. Yet Brouwer revolutionized it. (SEP has a 2013 article by Bridges)