



Memoryless determinacy of parity and mean payoff games: a simple proof[☆]

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Abstract

We give a simple, direct, and constructive proof of memoryless determinacy for parity and mean payoff games. First, we prove by induction that the finite duration versions of these games, played until some vertex is repeated, are determined and both players have memoryless winning strategies. In contrast to the proof of Ehrenfeucht and Mycielski, *Internat. J. Game Theory*, 8 (1979) 109–113, our proof does not refer to the infinite-duration versions. Second, we show that memoryless determinacy straightforwardly generalizes to infinite duration versions of parity and mean payoff games.

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1. Introduction

Parity games are infinite duration games played by two adversaries on finite leafless graphs with vertices colored by nonnegative integers. One of the players tries to ensure that the maximal vertex color occurring in the play infinitely often is *even*, while the other wants to make it *odd*. The problem of deciding the winner in parity games is polynomial time equivalent to the Rabin chain tree automata (or parity tree automata) nonemptiness, and to the model checking problem for the μ -calculus [6], one of the most expressive temporal logics of programs, expressively subsuming virtually

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all known such logics. For these reasons, parity games are of considerable importance and have been extensively studied by the complexity-theoretic, automata-theoretic, and verification communities.

One of the fundamental properties of parity games, which almost all decision algorithms rely upon, is the so-called *memoryless determinacy*. Vertices of every game can be partitioned into winning sets of both players, who possess positional winning strategies from all vertices in their winning sets. This means that for each vertex owned by a player, the player can decide in advance what to do if the play reaches that vertex, by deterministically selecting one of the outgoing edges independently of the history of the play. Moreover, revealing this positional strategy in advance is not a disadvantage.

Emerson [4] sketched the first memoryless determinacy proof for parity games¹ as early as in 1985. His proof is based on a (fairly complicated) simplification by Hossley and Rackoff [8] (relying on König's lemma) of Rabin's original proof [12] of the nonemptiness problem for Rabin automata. A later, more self-contained, determinacy proof by Emerson and Jutla [5] relies heavily on the μ -calculus, and is *non-constructive*. For example, the definition of a strategy in [5] uses properties of all paths in a binary tree, a set of continuum cardinality. Later McNaughton [10] and Zielonka [15] gave alternative proofs discussed below.

Today it is interesting to note that memoryless determinacy of parity games is a one-line consequence (using a simple reduction; see, e.g., Puri [11]) of the earlier more general result of Ehrenfeucht and Mycielski [2,3] on memoryless determinacy of the so-called *mean payoff games*.

Mean payoff games are also infinite duration games played by two adversaries on finite graphs, but with weighted edges. Players try to maximize/minimize the limit mean value of edge weights encountered during the play. It was proved by Ehrenfeucht and Mycielski [2,3] that every mean payoff game has a unique value v such that Player 0 can ensure a gain of at least v and Player 1 can ensure a loss of at most v , i.e., the games are *determined*. Furthermore, each player can secure this value by using a *positional (memoryless)* strategy.

The proof for mean payoff games given by Ehrenfeucht and Mycielski [3] relies upon a sophisticated *cyclic* interplay between *infinite* duration games and their *finite* counterparts. Proofs for infinite games rely on properties of finite games and vice versa. The authors asked whether it is possible to give a direct, rather than roundabout proof, a question we succeeded to answer affirmatively in this paper. Memoryless determinacy for mean payoff games was later shown constructively by Gurvich et al. [7], but their proof is rather involved, using estimates of norms of solutions to systems of linear equations, convergence to the limit, and precision arguments.

The purpose of this paper is to give a simple and direct proof that works uniformly for both parity and mean payoff games. Our proof does not involve any auxiliary constructions or machinery (like μ -calculus, or linear algebra, or limits), proceeds by elementary induction, and constructs positional strategies for more complicated games from strategies for simpler ones. Similar to [3], we rely on *finite duration* versions

¹ Actually for Rabin pairs automata, but easily transfers to parity games/automata.

of the games, played until the first vertex repetition. However, in contrast to [3], we completely avoid roundabout proofs of the properties of finite games using infinite ones and vice versa. Our proof is constructive, although the algorithm it provides is not intended to be very efficient. Due to the importance of parity games and mean payoff games, we feel that a straightforward and constructive proof of memoryless determinacy, common to both games, and without involving external powerful methods should be interesting and useful.

Two interesting memoryless determinacy proofs were given by McNaughton [10] and Zielonka [15], and both have very nice features. McNaughton studies a broad abstract class of games on *finite graphs* (including parity games), for which the winning conditions can be stated in terms of winning subsets of vertices. His proof provides a necessary and sufficient condition for such games to possess memoryless determinacy. This proof is constructive, but does not apply directly to mean payoff games, since the set of vertices visited infinitely often in a play does not uniquely determine its value. Zielonka [15] gives two simple and elegant proofs that work for parity games with a possibly *infinite* number of vertices, but not for mean payoff games. The first version of Zielonka's proof is constructive and uses induction on the number of colors and vertices (transfinite induction if there are infinitely many vertices). The second version is shorter but nonconstructive.

In contrast, our argument exploits structural similarities of parity and mean payoff games to give a uniform proof for both. It would be interesting to know whether our proof technique can be extended to more general classes of *discounted* and *simple stochastic games* [16]. Our current assumption on a winning condition, which allows to reduce infinite to finite duration games and conduct the uniform proof, is too strong, and should be relaxed to cover discounted and simple stochastic games.

The interest in parity games and mean payoff games is to a large extent motivated by their complexity-theoretic importance. For both games, the corresponding decision problems belong to the complexity class $\text{NP} \cap \text{coNP}$, but their PTIME membership status remains open. Much of the research in the complexity of solving these games relies on memoryless determinacy, e.g., [1,7,9,13,16]. Some papers rely essentially on memoryless determinacy, as [1,16], while others prove it independently, explicitly or implicitly [7].

1.1. Outline of the paper

Section 2 gives basic definitions concerning parity games. Theorem 3.1 of Section 3 shows determinacy of finite duration parity games, with strategies requiring memory. The proof is by elementary induction on the number of vertices in a finite tree. Section 4 provides intermediate results about the properties of positional strategies in finite duration games, needed in later sections. In Section 5 we prove by induction on the number of edges in the games, that finite duration parity games are solvable in positional strategies. This is the main theorem in the paper. We extend the proof to infinite duration parity games in Section 6. In Section 7 we show how the proof for parity games generalizes to yield memoryless determinacy of mean payoff games.

2. Parity games

We assume the standard definition of *parity games* (PGs). These are *infinite* duration adversary full information games played by Players 0 and 1 on *finite* directed leafless graphs $G = (V, E)$ with vertices colored by natural numbers. The vertices of the graph are partitioned into sets V_0 and V_1 , and every vertex has at least one outgoing edge (i.e., there are no leaves or sinks). The game starts in some vertex, and Player i chooses a successor when a play comes to a vertex in V_i . In this way, the players construct an infinite sequence of vertices, called a *play*. The parity of the *largest* color occurring infinitely often in this play determines the winner: even/odd is winning for Player 0/Player 1, respectively.

A (general) *strategy* of Player i is a function that for every finite prefix of a play, ending in a vertex $v \in V_i$, selects a successor of v (move of the player). A *positional strategy* for Player i is a mapping selecting a unique successor of every vertex $v \in V_i$. When a play comes to $v \in V_i$, Player i unconditionally selects the unique successor determined by the positional strategy, independently of the history of the play. Thus positional strategies are *memoryless*.

When considering a positional strategy σ of Player i , it is often useful to restrict to the graph G_σ obtained from G by removing all outgoing edges from vertices in V_i , except those used by σ .

3. Finite duration parity games

Together with the infinite duration parity games defined above, we also consider their *finite duration* counterparts, fundamental to our proofs. Such games are played on the same graphs as PGs, but only until the first loop is constructed. We first establish memoryless determinacy for finite duration games and then extend it to infinite duration ones.

A *finite duration parity game* (FPG) G_a starts in vertex a , and players alternate moves until some vertex v_l is visited for the second time. At this point the loop from v_l to v_l constructed during the play is analyzed, and the maximal vertex color c occurring on this loop determines the winner. If c is even, then Player 0 wins; otherwise, Player 1 wins.

FPGs are finite perfect information zero-sum games (the loser pays \$1 to the winner). By the general theorem of game theory (proved nonconstructively using Brouwer's or Kakutani's fixpoint theorems), every such game has a value. However, for the special case of FPGs, the proof of this fact is very simple and constructive. The argument is quite well known and can be attributed to Zermelo [14]. Nevertheless, to make the paper self-contained and to stress that probabilistic strategies² are not needed to decide a winner in a finite duration parity game, we give a direct proof here.

² Necessary, for example, in the "stone-paper-scissors" game.

Proposition 3.1. *The vertices of any FPG G can be partitioned into sets $W_0(G)$ and $W_1(G)$ such that Player i can win G_v when $v \in W_i(G)$.*

Definition 3.2. The set $W_i(G)$ in Proposition 3.1 is called the *winning set* for Player i .

Proof. Starting from every vertex v construct an AND-OR tree of all possible developments of an FPG starting at v . This tree is finite, with leaves corresponding to the first vertex repetitions on paths from the root. Mark those leaves with 0 or 1 corresponding to which player wins in the leaf (on the corresponding loop). Evaluate the root of the tree bottom-up by repeatedly using the rules: (1) if a vertex u of Player i has a successor marked i , then mark u with i ; (2) if all successors of a vertex u of Player i are marked $1 - i$, then mark u with $1 - i$. This evaluation uniquely determines the mark of the root, 0 or 1. \square

Note that although implementing a winning strategy according to the construction in the above proof does not need randomness, it requires memory to keep the history of the play in order to decide where to move at each step. In the following sections, based on Proposition 3.1, we show how to construct *positional* strategies.

4. Positional strategies in finite duration (parity) games

Finite duration parity games are slightly less intuitive and require different arguments compared to their infinite counterparts. For example, any finite prefix of an infinite play does not matter when determining the winner in an infinite game. In contrast, this prefix is essential in deciding the winner in a finite duration game, because its every vertex is a potential termination point of the game. Some other common infinite-case intuitions fail for finite games, and need some extra care.

This section confirms two basic intuitions about positional strategies in FPGs. The first one is an apparently obvious fact that you can win from a vertex provided you win from some of its successors. It is used to prove the second, demonstrating that when a player uses an optimal positional strategy, the play never leaves her winning set. In infinite games, both lemmas are one-line consequences of memoryless determinacy, but we stress that we need to prove these properties for finite games and without relying on determinacy.

Lemma 4.1. *In any FPG, if Player i has a positional strategy winning from some successor u of $v \in V_i$, then Player i has a positional strategy winning from v .*

(The straightforward construction “play from v to u and then follow the positional winning strategy from u ” is not completely obvious. Assume a positional winning strategy from u uses in v an edge different from (v, u) . The previous construction simply does not yield a *positional* strategy.)

Proof. Let us briefly discuss the recurring idea of the arguments we rely upon. Suppose Player i fixes a positional strategy σ , and consider the graph G_σ , where all other choices for Player i are removed. Then Player i wins from a vertex x iff Player $1 - i$ cannot force a play in G_σ from x into a simple loop losing for Player i .

Let σ be a positional strategy winning for Player i from some successor u of $v \in V_i$. If σ is winning also from v (which can be checked by inspecting the loops reachable from v in G_σ as explained above), the claim follows. Otherwise, there is a loop λ , losing for Player i , reachable by a path π from v in G_σ . We claim that Player $1 - i$ cannot force a play in G_σ from u to any vertex on π and λ , including v . Indeed, the opposite would imply that Player i loses from u . Change σ only at v , obtaining σ' with $\sigma'(v) = u$. Player i still wins from u with σ' , since the set of plays in G'_σ from u remains the same as in G_σ , and exactly the same loops can be forced from v by Player $1 - i$. \square

Important observation: We pause here to make an important observation. The argument in the proof of Lemma 4.1 actually applies to the whole class of finite duration games that, like FPGs: (1) stop as soon as some vertex is first revisited, (2) for which the winner is determined by the sequence of vertices on the resulting simple loop, and (3) independently of the initial vertex the loop is traversed from. Condition 3 is not satisfied, e.g., for finite versions of discounted mean payoff games or simple stochastic games [16].

This class includes, in particular, the finite decision version of mean payoff games, considered in Section 7. We encourage the reader to verify that all subsequent proofs in this section and Section 5 apply for this general class of games. Actually, our proof of memoryless determinacy of finite mean payoff games in Section 7.2 relies on this observation and thus recycles the work done for parity games.

The next lemma puts the (yet unproved) assumption “if both players win by positional strategies from every vertex in their respective winning³ sets” in the premise. Under this assumption it shows that no play consistent with such a winning positional strategy leaves the player’s winning set.

Lemma 4.2. *In any FPG, suppose σ_0 and σ_1 are positional strategies for Player 0 and 1 such that Player i by using σ_i wins every game that starts in $W_i(G)$, for $i \in \{0, 1\}$. Then all plays starting in $W_i(G)$ and played according to σ_i stay in $W_i(G)$.*

Proof. Consider a game on the graph G_{σ_i} starting in some vertex from $W_i(G)$. In this game, we can assume that all vertices belong to Player $1 - i$ because there are no choices in Player i ’s vertices. By Lemma 4.1, any edge (u, v) with $v \in W_{1-i}(G)$ must also have $u \in W_{1-i}(G)$, so there are no edges from $W_i(G)$ to $W_{1-i}(G)$ in G_{σ_i} . \square

This lemma will be extensively used in the inductive proof of the main theorem in the next section, where its premise will be guaranteed to hold by inductive assumption.

³ Winning by some, maybe nonpositional strategy.

5. Memoryless determinacy of finite duration parity games

In this section we prove the main theorem of the paper, which implies memoryless determinacy of the infinite duration parity games (Section 6). The proof itself does not use any reference to the infinite games. In Section 7 we extend this theorem to finite and infinite duration mean payoff games. Actually, the proof of Theorem 5.1 is not modified, we just show how to adjust winning conditions so as to reuse the theorem and its proof as they are.

Theorem 5.1. *In every FPG G , there are positional strategies σ_0 of Player 0 and σ_1 of Player 1 such that Player i wins by using σ_i whenever a play starts in $W_i(G)$, no matter which strategy Player $1-i$ applies.*

Proof. The proof is by induction on the number $|E|-|V|$ of choices of both players. We stress once again that the proof applies to a more general class of games, as observed in Section 4, and therefore will be reused to demonstrate memoryless determinacy of the decision version of finite mean payoff games in Section 7.2.

The *base case* is immediate: if $|E|=|V|$ then there are no vertices with a choice, and all strategies are positional.

For the *inductive step* we split into three cases, depending on whether there are vertices x from which the player who owns x can win. Let $V_i^* \subseteq V_i$ (for $i=0,1$) be the sets of vertices of Player i with outdegree more than 1. Assume the theorem holds whenever $|E|-|V|<n$, and consider $|E|-|V|=n$.

Case 1: $V_0^* \cap W_0(G) \neq \emptyset$. Take $x \in V_0^* \cap W_0(G)$ and let e be an edge leaving x such that Player 0 can win G_x after selecting e in the first move. (The rest of the Player 0 strategy does not need to be positional.) Consider the game G' in which we remove all edges leaving x , except e . By inductive assumption, there are positional strategies σ_i such that Player i applying σ_i wins any play in G' that starts in $W_i(G')$. These strategies are also positional strategies in G . We will prove that σ_i is winning for Player i in G if the play starts in a vertex from $W_i(G')$.

Suppose $v \in W_0(G')$ and Player 0 uses σ_0 in G_v . Any play in G_v following σ_0 was also possible in G'_v . Since all such plays are winning for Player 0 in G'_v , all of them are also winning for Player 0 in G_v . Therefore, $W_0(G') \subseteq W_0(G)$.

Now assume $v \in W_1(G')$ and Player 1 uses σ_1 in G_v . Notice that $x \in W_0(G')$, since there is a winning strategy that uses e and the game terminates as soon as x is revisited. By Lemma 4.2, whenever Player 1 uses σ_1 , no play in G'_v , hence in G_v , can ever reach x , because G'_v and G_v differ only in edges *leaving* x . But any play in G_v according to σ_1 not reaching x was possible also in G'_v , so Player 1 wins any such play in G_v . Hence, $W_1(G') \subseteq W_1(G)$.

Since $W_i(G') \subseteq W_i(G)$, and $W_0(G')$, $W_1(G')$ form a partition, we thus showed that $W_i(G) = W_i(G')$ and Player i wins G_v using σ_i , for any $v \in W_i(G)$.

Case 2: $V_1^* \cap W_1(G) \neq \emptyset$ and *Case 1 does not apply*. The proof is symmetrical to Case 1.

Case 3: $V_0^* \cap W_0(G) = V_1^* \cap W_1(G) = \emptyset$. Since only Player i may have choices in $W_{1-i}(G)$, we will assume that all vertices in $W_i(G)$ belong to V_{1-i} , that is, $W_i(G) = V_{1-i}$

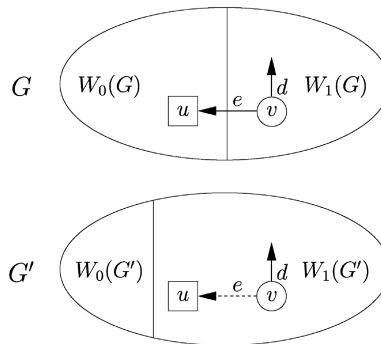


Fig. 1. If we remove e from G , then u changes from losing to winning for Player 0.

for $i \in \{0, 1\}$. We may also assume that one of the players has choices; otherwise the base case applies.

We first prove that there are no edges between the winning sets. Suppose, towards a contradiction, that there is an edge e from $v \in W_1(G)$ to $u \in W_0(G)$. Fig. 1 depicts this situation, where round and square vertices belong to Player 0 and 1, respectively (the case of an edge back is symmetric).

Although u is losing for Player 1, there must be a way for Player 1 to win a play that starts in v even if Player 0 moves by e to u . This can *only* be done if Player 1 can force the play back to v , because any other winning play for Player 1 from v via u would be a winning play from u as well. However, Player 1 cannot win if a play starts in u . Therefore, if the game starts in u , and is forced by Player 1 to v , Player 0 must be able to do something else in order to win, rather than selecting e . Thus there must be another edge d leaving v .

Now create the game G' by removing the edge e . Since Player 0 has less choices, we must have $W_0(G') \subseteq W_0(G)$. By inductive assumption, there are positional strategies σ_0 and σ_1 , winning from $W_0(G')$ and $W_1(G')$, respectively. Since all vertices in $W_0(G')$ can be assumed to belong to V_1 , Lemma 4.2 implies that there are no edges leaving $W_0(G')$. This in turn means that u belongs to $W_1(G')$. Otherwise, Player 0 could have won from v in G by following e , because the play would have never left $W_0(G')$, and any loop formed would have been winning for Player 0.

When e was removed, vertex u turned from losing to winning for Player 1. This implies that in G' , Player 1 must force the play from u to v in order to win, since all other plays would have been possible in G as well, and all were losing for Player 1. In G , however, no matter how Player 1 forced the play from u to v , Player 0 could win by using d and some strategy for the remaining play. All these plays are possible in G' as well, so Player 1 cannot win from u in G' . This contradiction shows that there can be no edge e from $W_1(G)$ to $W_0(G)$. By symmetric reasoning, there are no edges from $W_0(G)$ to $W_1(G)$.

Now, since there are no edges between the winning sets, and both players lack choices within their own winning sets, any positional strategy is optimal, i.e., winning from all vertices in the player's winning set. \square

6. Extension to infinite duration parity games

We now show that a positional strategy that wins an FPG starting in vertex v also wins, for the same player, the infinite duration parity game starting in v on the same game graph.

Let σ be a winning strategy (positional or not) of Player i (for $i \in \{0, 1\}$) in the FPG starting from v . No matter what the opponent does, the first revisit to a previously visited vertex v_l guarantees that the maximal color on the loop from v_l to v_l is winning for Player i . Now forget what happened on the path from v_l to v_l , assume v_l is visited for the first time, and let the FPG develop further. The next time some vertex v'_l (not necessarily equal to v_l) is revisited, we also know that the maximal color on the loop from v'_l to v'_l is winning for Player i . Again, forget what happened on the path from v'_l to v'_l , assume v'_l is visited for the first time, and let the FPG develop further. In this way, using the winning strategy for Player i , we construct an infinite sequence of loops and the corresponding infinite sequence of maximal colors $S = \{c_i\}_{i=1}^{\infty}$ winning for Player i . It follows that the maximal color hit in the infinite game infinitely often is the maximum appearing in S infinitely often. This also means that the winning sets of the players in the infinite and finite games coincide.

The following theorem makes the above argument formal. It also establishes the converse, that positional winning strategies in infinite parity games are winning in finite ones.

Theorem 6.1. *A positional strategy σ of Player i wins in an infinite duration PG G_v if and only if it wins in the corresponding FPG.*

Proof. We will show that σ wins the infinite game if and only if the highest color on every simple loop reachable from v in G_σ is winning for Player i . The latter is equivalent to σ winning the finite game.

If G_σ contains a loop reachable from v with a highest color losing for Player i , then σ is losing: Player $1 - i$ can go to this loop and stay in it forever.

Conversely, suppose G_σ does not contain any loops with losing highest color reachable from v . We will prove that there is no infinite path starting from v in G_σ on which the highest color appearing infinitely often is losing for Player i . Assume, towards a contradiction, that there is such a path p on which the highest color occurring infinitely often is c . There must be some vertex u of color c that appears infinitely often on p . But between any two such appearances of u on the path, all other vertices on some simple loop containing u must appear. Among these vertices, at least one should have a winning color higher than c , by assumption. This means that some winning color higher than c appears infinitely often on p , contradicting the assumption. \square

As a direct consequence, the memoryless determinacy of Theorem 5.1 holds also for infinite duration parity games.

Corollary 6.2. *In every PG G , there are positional strategies σ_0 of Player 0 and σ_1 of Player 1 such that Player i wins by using σ_i whenever a play starts in $W_i(G)$, no matter which strategy Player $1 - i$ applies.*

7. Extension to mean payoff games

This section shows how the memoryless determinacy proof for parity games extends to mean payoff games.

7.1. Finite and infinite duration mean payoff games

Mean payoff games [3,7,16] are similar to parity games. Let $V = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, $E \subseteq V \times V$ (where, for each $u \in V$, there is some v with $(u, v) \in E$), and $c: E \rightarrow \mathbb{R}$. Define the game graph $\Gamma = \Gamma(V_0, V_1, E, c)$. Starting in some predefined vertex v_0 , the two players move by selecting edges from their respective vertex sets in the same way as in parity games. This yields an infinite play (sequence of vertices) $p = v_0 v_1 v_2 \dots$. Player 0 wants to maximize

$$v_0(p) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} c(v_i, v_{i+1})$$

and Player 1 wants to minimize

$$v_1(p) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} c(v_i, v_{i+1}).$$

Analogously to the finite version of parity games, we define *finite duration mean payoff games* (FMPGs). Like in FPGs, a play starts in the initial vertex v_0 and ends as soon as a loop is formed. The value $v(p)$ of the play $p = v_0 v_1 \dots v_m \dots v_n$, where $v_m = v_n$, is the mean value of the edges on this loop:

$$v(p) = \frac{1}{n - m} \sum_{i=m}^{n-1} c(v_i, v_{i+1}).$$

Player 0 wants to maximize this value and Player 1 wants to minimize it.

FMPGs are, like FPGs, finite, zero-sum, perfect information games, and are therefore determined. Also like FPGs, FMPGs are a special case of such games, for which this fact can be proved in an elementary way.

Proposition 7.1. *For every FMPG starting in any vertex u , there is a value $v(u)$ such that Player 0 can ensure a gain of at least $v(u)$, and Player 1 can ensure a loss of at most $v(u)$, independently of the opponent's strategy.*

Proof. Similar to the proof of Proposition 3.1, consider the finite tree of all possible plays from u . Assign to each leaf the value of the play it represents. Let all internal vertices corresponding to choices of Player 0 be MAX-vertices, and let the choices of Player 1 be MIN-vertices. The root of the resulting MIN-MAX-tree can be straightforwardly evaluated in a bottom-up fashion, and its value is the value $v(u)$ of the game starting from u . \square

The following lemma, proved in [3], using only elementary means, shows that a strategy in an FMPG G yields a strategy in the corresponding MPG Γ , guaranteeing

the same value. Thus it is enough to show that there are optimal positional strategies in FMPGs.

Lemma 7.2 (Ehrenfeucht-Mycielski [3]). *A strategy σ of Player i in an FMPG G can be modified to a strategy $\tilde{\sigma}$ in the corresponding MPG Γ such that the following properties hold.*

- (1) *If σ secures the value v in G , then $\tilde{\sigma}$ secures v in Γ .*
- (2) *If σ is positional, then $\tilde{\sigma} = \sigma$ so $\tilde{\sigma}$ is also positional. \square*

7.2. Main theorem for the decision version of finite mean payoff games

Consider the *decision* version of finite duration mean payoff games, denoted FMPG(D), in which we are only interested in whether the value of a play is greater than some threshold t , not in the actual value. The winning condition for Player 0 is either $v(p) > t$ or $v(p) \geq t$. This allows us to define *winning sets* similarly to the case of finite duration parity games. Say that $u \in W_0(G)$ if Player 0 has a strategy that ensures a value $v(u) \geq t$, and $u \in W_1(G)$ otherwise.

As was observed in Sections 4 and 5, the proofs of Lemmas 4.1, 4.2, and of Theorem 5.1 work without any modifications for a broader class of games satisfying the following.

- Assumptions.** (1) A play on a game graph starts in a vertex and stops as soon as a loop is formed, and
- (2) The winner is determined by the sequence of vertices on the loop, modulo cyclic permutations.

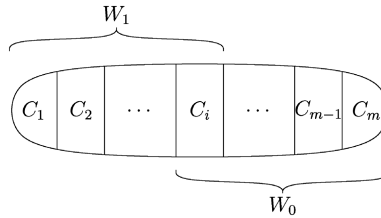
These assumptions on the winning condition are sufficient to prove Theorem 5.1, and Lemmas 4.1 and 4.2 upon which it depends, as can be readily verified by inspecting their proofs. It clearly holds for FMPG(D)s, where a sequence of vertices *uniquely* determines a sequence of edges, because there are no multiple edges, and their average cost determines the winner. Therefore, we obtain the following memoryless determinacy result for the decision version of finite mean payoff games.

Theorem 7.3. *In every FMPG(D), each player has a positional strategy that wins from all vertices in the player's winning set. \square*

By Lemma 7.2, this result extends to the infinite duration mean payoff games.

7.3. Ergodic partition theorem for mean payoff games

In this section we reinforce Theorem 7.3 by proving that each vertex v in an FMPG has a value, which both players can secure by means of positional strategies whenever

Fig. 2. The ergodic partitions of G .

a play starts in v . Moreover, the same pair of strategies can be used independently of the starting vertex. More formally:

Theorem 7.4 (Memoryless determinacy and ergodic partition). *Let G be an FMPG and $\{C_i\}_{i=1}^m$ be a partition (called ergodic) of its vertices into classes with the same value x_i , as given by Proposition 7.1. There are positional strategies σ_0 and σ_1 for Player 0 and 1 with the following properties:*

if the game starts from a vertex in C_i , then σ_0 secures a gain $\geq x_i$ for Player 0, and σ_1 secures a loss $\leq x_i$ for Player 1.

Moreover, Player 0 has no vertices with outgoing edges leading from C_i to C_j with $x_i < x_j$, and Player 1 has no vertices with outgoing edges leading from C_i to C_j with $x_i > x_j$.

Proof. By Proposition 7.1, there exist values $x_1 < x_2 < \dots < x_m$ and a partition C_1, C_2, \dots, C_m such that for every starting vertex $u \in C_i$ of an FMPG G both players ensure for themselves (possibly by nonpositional strategies) the value x_i . Now, for every value x_i solve two FMPG(D) problems (using Theorem 7.3), as shown in Fig. 2:

- (1) find the winning set W_0 and corresponding strategy σ_0 of Player 0 securing a gain $\geq x_i$ when a play starts in W_0 .
- (2) find the winning set W_1 and corresponding strategy σ_1 of Player 1 securing a loss $\leq x_i$ when a play starts in W_1 .

Consider $W_0 \cap W_1$. In this (nonempty) set both Player 0 can secure a gain $\geq x_i$ by means of σ_0 , and Player 1 can secure a loss $\leq x_i$ by means of σ_1 . In other words, $W_0 \cap W_1 = C_i$.

By Lemma 4.2, any play starting from W_0 always stays in this set, when Player 0 uses σ_0 (Player 1 has no edges out of it), and symmetrically for W_1 .

We repeat the argument above for all values x_1, \dots, x_m getting winning positional strategies σ_0^i, σ_1^i . Using the property from the preceding paragraph we merge all these strategies into positional strategies σ_0 and σ_1 for Player 0 and Player 1 as stated in the theorem. Simply define σ_0 as coinciding with σ_0^i on the set of vertices $V_0 \cap C_i$, and similarly for σ_1 . \square

By Lemma 7.2, the same results on memoryless determinacy and ergodic partition from Theorem 7.4, hold also for (infinite duration) mean payoff games. Moreover, it is

easy to see that these results hold (with the same proof) in the version of parity games where Player 0 and 1 not only want to win with *some* even/odd color, but want to win with *highest possible even/odd* color. This version of parity games is especially suitable for solving by means of a randomized subexponential algorithm described in [1], as well as by an iterative strategy improvement algorithm from [13].

8. Conclusions

We have presented a new unified proof of the well-known memoryless determinacy results for parity and mean payoff games on finite graphs. There are several previous proofs, but we nonetheless think our proof is useful since it combines several nice properties. It is simple and constructive, providing an easy introduction to the field. Relying only on elementary methods, it illustrates that proving the basic properties of infinite games does not need to attract external notions of limits and approximation. The distinctive feature of our proof is that we first establish memoryless determinacy for the finite duration versions of the games and then extend it to infinite duration. As a consequence, we avoid cyclic, roundabout proofs, as in Ehrenfeucht–Mycielski [3], thus answering positively their question whether one could avoid circularity proving determinacy of infinite and finite duration mean payoff games. Our proof indicates that in this respect finite duration parity and mean payoff games are “more fundamental”, directly implying memoryless determinacy for their infinite duration counterparts. Furthermore, by applying to both parity and mean payoff games, the proof stresses the structural similarities between both games.

Can our proof be extended for more general classes of games including *discounted payoff games* and *simple stochastic games* [16]? Our current assumptions on the winning condition (see Section 7.2) do not apply to these games. Discounted payoff games can be formulated in a finite-duration version, but Lemmas 4.1 and 4.2 do not hold for them. We do not know whether there is a finite-duration version of simple stochastic games, and although some version of the lemmas hold in the infinite version, it is unclear whether this suffices for the proof. Nevertheless, we feel that the structures of these games have relevant features in common with parity and mean payoff games. It would be interesting to know whether memoryless determinacy can be proved uniformly for all four games, with our proof technique or another.

References

- [1] H. Björklund, S. Sandberg, S. Vorobyov, A discrete subexponential algorithm for parity games, in: H. Alt, M. Habib (Eds.), 20th Internat. Symp. on Theoretical Aspects of Computer Science, STACS'2003, Lecture Notes in Computer Science, Vol. 2607, Springer, Berlin, 2003, pp. 663–674. Full preliminary version: TR-2002-026, Department of Information Technology, Uppsala University, September 2002.
- [2] A. Ehrenfeucht, J. Mycielski, Positional games over a graph, Notices Amer. Math Soc. 20 (1973) A-334.
- [3] A. Ehrenfeucht, J. Mycielski, Positional strategies for mean payoff games, Internat. J. Game Theory 8 (1979) 109–113.

- [4] E.A. Emerson, Automata, tableaux, and temporal logics, in: R. Parikh (Ed.), Proc. Workshop on Logic of Programs, Lecture Notes in Computer Science, Vol. 193, Springer, Berlin, 1985, pp. 79–87.
- [5] E.A. Emerson, C.S. Jutla, Tree automata, μ -calculus and determinacy, in: Ann. IEEE Symp. on Foundations of Computer Science, 1991, pp. 368–377.
- [6] E.A. Emerson, C. Jutla, A.P. Sistla, On model-checking for fragments of μ -calculus, in: C. Courcoubetis (Ed.), Computer Aided Verification, Proc. 5th Internat. Conf., Lecture Notes in Computer Science, Vol. 697, Springer, Berlin, 1993, pp. 385–396.
- [7] V.A. Gurevich, A.V. Karzanov, L.G. Khachiyan, Cyclic games and an algorithm to find minimax cycle means in directed graphs, *USSR Comput. Math. & Math. Phys.* 28 (5) (1988) 85–91.
- [8] R. Hossley, C. Rackoff, The emptiness problem for automata on infinite trees, in: 13th Symp. on Automata and Switching Theory, 1972, pp. 121–124.
- [9] M. Jurdziński, Small progress measures for solving parity games, in: H. Reichel and S. Tison (Eds.), 17th STACS, Lecture Notes in Computer Science, Vol. 1770, Springer, Berlin, 2000, pp. 290–301.
- [10] R. McNaughton, Infinite games played on finite graphs, *Ann. Pure Appl. Logic* 65 (2) (1993) 149–184.
- [11] A. Puri, Theory of hybrid systems and discrete events systems, Ph.D. Thesis, EECS University, Berkeley, 1995.
- [12] M. Rabin, Decidability of second order theories and automata on infinite trees, *Trans. Amer. Math. Soc.* 141 (1969) 1–35.
- [13] J. Vöge, M. Jurdziński, A discrete strategy improvement algorithm for solving parity games, in: E.A. Emerson and A.P. Sistia (Eds.), CAV'00: Computer-Aided Verification, Lecture Notes in Computer Science, Vol. 1855, Springer, Berlin, 2000, pp. 202–215.
- [14] E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiele, in: E.W. Hobson, A.E.H. Love (Eds.), Proc. 5th Internat. Congr. Mathematics, Vol. 2, Cambridge, 1913, pp. 501–504.
- [15] W. Zielonka, Infinite games on finitely coloured graphs with applications to automata on infinite trees, *Theor. Comput. Sci.* 200 (1998) 135–183.
- [16] U. Zwick, M. Paterson, The complexity of mean payoff games on graphs, *Theor. Comput. Sci.* 158 (1996) 343–359.