Logics of Programs	**Homework 4**
CS686 Spring 2001	**Solutions Posted Thursday, April 19**

1. Prove in KAT that the following statements are equivalent:

   (a) $pb = bp$

   (b) $p\bar{b} = \bar{b}p$

   (c) $bp\bar{b} + \bar{b}pb = 0$.

2. We were speculating in class about the equivalence between the equational theory and the Hoare theory of KAT (formulas of the form $p = 0 \rightarrow q = 0$).

   (a) I had conjectured in class that $\vDash p \leq q$ iff REL $\vDash q = 0 \rightarrow p = 0$. As James Cheney pointed out, this is false. Give a counterexample.

   (b) Prove the following revised version: $\vDash p \leq q$ iff REL $\vDash uqv = 0 \rightarrow upv = 0$, where u and v are atomic programs that do not occur in p or q.

3. Let $\varphi[X/\psi]$ denote the result of substituting ψ for all free occurrences of X in φ, assuming that bound variables in φ are renamed so that no free variable in ψ is captured as a result of the substitution. Show that the following formulas are equivalent:

   (a) $\mu X.\mu Y.\varphi$

   (b) $\mu Y.\mu X.\varphi$

   (c) $\mu Z.(\varphi[X/Z][Y/Z])$.

4. (a) Let $\sigma : \{X, Y, \ldots\} \rightarrow 2^K$ be an environment (valuation of propositional variables) over a Kripke frame K. Prove that if the map $A \mapsto \sigma[X/A](\varphi)$ is continuous, then $\sigma(\mu X.\varphi) = \sigma(\omega X.\varphi)$.

   (b) Prove that in any Kripke frame, $s \vDash \mu X.[a]X$ iff there are no infinite a-paths out of s.

   (c) Construct for any ordinal α a Kripke frame $\mathcal{K}$ with a state s for which $\mathcal{K}, s \vDash \mu X.[a]X$ but $\mathcal{K}, s \nvDash \beta X.[a]X$ for any $\beta < \alpha$.

5. The modal μ-calculus can be used to express and reason about termination in the presence of simple fairness constraints. Suppose we are given a nondeterministic program for which we know there is some scheduling mechanism that resolves the nondeterminism, but we don't know how it works; it is just a black box. However, we are told that it is *fair* in the sense that every request that becomes enabled is eventually satisfied by the scheduler. We wish to show that the program terminates under those conditions. In other words,

there may be infinite computation paths, but all such computation paths would be the result of an unfair schedule, which the scheduler would never allow to occur.

This could be formalized as follows. There are a finite set of requests $p_1, \ldots, p_n$, which are just atomic propositions, and a finite set of actions $a_1, \ldots, a_n$ that satisfy those requests, which are just atomic modalities. A path in a model is *unfair* if there is a state t occurring somewhere along the path such that p_i is true at t and every state following t on the path, but none of the edges along the path after t are labeled a_i.

The program is *fairly terminating* from state s if every infinite path out of s is unfair. Give a formula of the μ-calculus that expresses this.