Logics of Programs	**Homework 2 Solutions**
CS686 Spring 2001	**Monday, February 26, 2001**

1. (a) First we show $x \vee (x \wedge y) = x \wedge (x \vee y) = x$.

$$\begin{aligned} x \vee (x \wedge y) &= (x \vee (x \wedge y)) \wedge 1 \\ &= (x \vee (x \wedge y)) \wedge (x \vee \neg x) \\ &= x \vee (x \wedge y \wedge \neg x) \\ &= x \vee 0 \\ &= x. \end{aligned}$$

By a symmetric argument, $x \wedge (x \vee y) = x$. Then

$$\begin{aligned} a \vee b = b &\;\Rightarrow\; a = a \wedge (a \vee b) = a \wedge b \\ a \wedge b = a &\;\Rightarrow\; b = b \vee (a \wedge b) = b \vee a. \end{aligned}$$

(b) Lemma: If $a \wedge b = 0$ and $a \vee b = 1$, then $a = \neg b$.
Proof: Using the assumption $a \vee b = 1$,

$$\begin{aligned} \neg b &= (a \vee b) \wedge \neg b \\ &= (a \wedge \neg b) \vee (b \wedge \neg b) \\ &= (a \wedge \neg b) \vee 0 \\ &= a \wedge \neg b. \end{aligned}$$

By a symmetric argument using the other assumption $a \wedge b = 0$, we get $\neg b = a \vee \neg b$. By (a), $a = a \wedge \neg b$, therefore $a = \neg b$.

By the lemma, to prove the De Morgan law $\neg(x \vee y) = \neg x \wedge \neg y$, it suffices to show $x \vee y \vee (\neg x \wedge \neg y) = 1$ and $(x \vee y) \wedge \neg x \wedge \neg y = 0$. These follow directly from distributivity and the 0-1 axioms. The proof of $\neg(x \wedge y) = \neg x \vee \neg y$ is symmetric.

The double negation law $\neg\neg x = x$ follows immediately from the lemma and the axioms $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

2. We must show $\leq$ is reflexive, antisymmetric, and transitive. Reflexive: $a = a \wedge a$, so $a \leq a$. Antisymmetric: if $a \leq b$ and $b \leq a$, then $a \wedge b = a$ and $b \wedge a = b$, so by commutativity of $\wedge$, $a = b$. Transitive: if $a \leq b \leq c$, then $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$, therefore $a \leq c$. The bottom element is 0 and the top is 1, since $0 = 0 \wedge a$ and $a = a \wedge 1$ for any a.

3. (a) The inverse image of any filter in A under a homomorphism $h : B \to A$ is a filter in B. In particular, the inverse image of the trivial filter $\{1\}$ is a filter in B. For example, if F is a filter in A, then

$$\begin{aligned} x, y \in h^{-1}(F) &\Rightarrow h(x), h(y) \in F \\ &\Rightarrow h(y) \wedge h(y) \in F \\ &\Rightarrow h(x \wedge y) \in F \\ &\Rightarrow x \wedge y \in h^{-1}(F). \end{aligned}$$

The other properties of filters are argued similarly, using the facts that h is monotone with respect to the natural order in A and B, $h(0) = 0$, and $h(1) = 1$.

Conversely, given any filter F, define $x \equiv y$ if $x \leftrightarrow y \in F$, where $x \leftrightarrow y \stackrel{\text{def}}{=} (x \wedge y) \vee (\neg x \wedge \neg y)$ (intuitively: two propositions φ and ψ are propositionally equivalent iff the proposition $\varphi \leftrightarrow \psi$ is a tautology). The relation $\equiv$ is a Boolean algebra congruence on B; that is, an equivalence relation that respects all the Boolean algebra operations in the sense that $x \equiv y \Rightarrow x \vee z \equiv y \vee z$, etc. This allows us to form the *quotient algebra* $B/\equiv$ whose elements are the congruence classes $[x] \stackrel{\text{def}}{=} \{y \mid x \equiv y\}$ with operations $[x] \vee [y] \stackrel{\text{def}}{=} [x \vee y]$, etc., and these operations are well-defined. The filter F is the inverse image of $\{1\}$ under the natural homomorphism $x \mapsto [x] : B \to B/\equiv$.

(b) This follows immediately from part (a) and the property of ultrafilters that for all $x \in B$ and ultrafilters u, either $x \in u$ or $\neg x \in u$.

4. The algebra B_Y has the same operations as B but restricted to Y. Thus the top element of B_Y is Y (which is an element of B_Y, but not necessarily an element of B) and negation defined by set complementation with respect to Y. To show that the map $x \to x \cap Y$ is a Boolean algebra homomorphism, we must show that it respects all the Boolean operations.

$$\begin{aligned} (x \cap y) \cap Y &= (x \cap Y) \cap (y \cap Y) \\ (x \cup y) \cap Y &= (x \cap Y) \cup (y \cap Y) \\ (X - x) \cap Y &= Y - x \\ \varnothing \cap Y &= \varnothing \\ X \cap Y &= Y. \end{aligned}$$

The kernel of this map is $\{x \in B \mid x \cap Y = \varnothing\}$.

5. (a) By the Stone representation theorem, any Boolean algebra is isomorphic to a set-theoretic Boolean algebra on a set X, and this set-theoretic algebra is a subalgebra of the atomic Boolean algebra 2^X. The atoms of 2^X are the singleton sets $\{s\}$ for $s \in X$.

(b) Let $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$ be countable atomless Boolean algebras. We will construct an isomorphism between A and B as a union of finite partial isomorphisms. This construction is commonly known as a *back-and-forth argument*.

Start with the empty relation at stage 0. Now suppose after stage n we have constructed an isomorphism $h_n : A_n \to B_n$ between finite subalgebras of A and B. (In our construction, these subalgebras are generated by sets of size n and have equal cardinality at most 2^{2^n}.) If n is even, in the next stage we will pick the next unmatched element $a \in A$ (that is, the element $a_i \in A - A_n$ with minimum index i) and try to find an appropriate $b \in B - B_n$ to pair with it. If n is odd, we will take the next unmatched $b \in B - B_n$ and try to find an appropriate $a \in A - A_n$ to pair with it. Thus every element of A eventually becomes paired with an element of B and vice versa. We must pick the b at even stages and the a at odd stages to maintain the partial isomorphism.

At even stages, given $a \in A - A_n$, let $b \in B - B_n$ be such that for each atom x of A_n, $x \wedge a = 0 \Leftrightarrow h_n(x) \wedge b = 0$ and $x \wedge \neg a = 0 \Leftrightarrow h_n(x) \wedge \neg b = 0$. Such a b can always be found: since B is atomless, every atom $h_n(x)$ of B_n can be expressed as the join of two disjoint nonzero elements of B. Define A_{n+1} to be the subalgebra of A generated by $A_n \cup \{a\}$ and B_{n+1} to be the subalgebra of B generated by $B_n \cup \{b\}$. Extend h_n to $h_{n+1} : A_{n+1} \to B_{n+1}$ by assigning $h_{n+1}(a) = b$. The atoms of A_{n+1} are the nonzero elements of the form $x \wedge a$ or $x \wedge \neg a$ for x an atom of A_n, and $x \wedge a = 0$ iff $h_{n+1}(x \wedge a) = h_n(x) \wedge b = 0$ and $x \wedge \neg a = 0$ iff $h_{n+1}(x \wedge \neg a) = h_n(x) \wedge \neg b = 0$. Since every element of A_{n+1} is uniquely represented as a disjoint sum of atoms of A_{n+1}, it follows that h_{n+1} is an isomorphism $A_{n+1} \to B_{n+1}$.

The argument for odd stages is symmetric. The isomorphism $A \to B$ is the limit of this construction.

6. (a) Let T be the set of truth assignments to X. There are 2^n elements of T and 2^{2^n} subsets of T. We argue that $\mathcal{F}_{X,\varnothing}$ is isomorphic to the Boolean algebra 2^T under the map $h([\varphi]_\varnothing) = \{\text{truth assignments to } X \text{ satisfying } \varphi\}$. This map is well-defined and one-to-one, since $\varphi \equiv_\varnothing \psi$ iff $\varphi \leftrightarrow \psi$ is a tautology iff φ and ψ are satisfied by the same set of truth assignments. It is also onto, since every set of truth assignments is represented by some formula. Moreover, it is easily verified that it is a Boolean algebra homomorphism:

$$\begin{aligned} & h([\varphi]_\varnothing \vee [\psi]_\varnothing) \\ &= h([\varphi \vee \psi]_\varnothing) \\ &= \{\text{truth assignments satisfying } \varphi \vee \psi\} \\ &= \{\text{truth assignments satisfying } \varphi\} \cup \{\text{truth assignments satisfying } \psi\}, \end{aligned}$$

etc.

Note that the set-theoretic algebra 2^T is essentially the same as the algebra that would be produced from $\mathcal{F}_{X,\varnothing}$ under the Stone construction, since the truth assignments to X and ultrafilters of $\mathcal{F}_{X,\varnothing}$ are in one-to-one correspondence.

(b) Suppose X is infinite. Let φ be a consistent propositional formula over X; thus $[\varphi]_\varnothing \neq \mathbf{0}$ in $\mathcal{F}_{X,\varnothing}$. Let x be an element of X not occurring in φ. Then $(x \wedge \varphi) \to \varphi$

is a tautology, but $\varphi \to (x \wedge \varphi)$ is not a tautology, because there is a truth assignment under which φ is true and x is false. Thus $[\varphi \wedge x] < [\varphi]$ in $\mathcal{F}_{X,\varnothing}$. By a similar argument, $[\varphi \wedge \neg x] < [\varphi]$. Since $[\varphi]$ can be decomposed into a disjoint sum $[\varphi \wedge x] \vee [\varphi \wedge \neg x]$ of strictly smaller elements, it is not an atom. Since φ was arbitrary, $\mathcal{F}_{X,\varnothing}$ is atomless.

(c) The natural homomorphism is $h([\varphi]_\varnothing) = [\varphi]_\Gamma$. This is well-defined, since $\varphi \equiv_\varnothing \psi$ implies $\varphi \equiv_\Gamma \psi$.

(ii) $\Leftrightarrow$ (iii): $[\varphi]_\varnothing$ is in the filter of $\mathcal{F}_{X,\varnothing}$ generated by the $\equiv_\varnothing$-classes of elements of Γ iff there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that $[\psi_1]_\varnothing \wedge \ldots \wedge [\psi_n]_\varnothing \le [\varphi]_\varnothing$. This occurs iff $[\psi_1 \wedge \ldots \wedge \psi_n]_\varnothing \le [\varphi]_\varnothing$, which says that $\psi_1 \wedge \ldots \wedge \psi_n \to \varphi$ is a tautology. This implies that φ is a logical consequence of $\{\psi_1, \ldots, \psi_n\}$ and hence of Γ. Conversely, if $\Gamma \vDash \varphi$, then by compactness there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\{\psi_1, \ldots, \psi_n\} \vDash \varphi$, thus by the deduction theorem $\psi_1 \wedge \ldots \psi_n \to \varphi$ is a tautology.

(iii) $\Leftrightarrow$ (i): $\Gamma \vDash \varphi$ iff $\Gamma \vDash \varphi \leftrightarrow \mathbf{1}$ iff $h([\varphi]_\varnothing) = [\varphi]_\Gamma = [\mathbf{1}]_\Gamma = \mathbf{1}$ iff $h([\neg\varphi]_\varnothing) = \neg h([\varphi]_\varnothing) = \mathbf{0}$.

(d) Let B be an arbitrary Boolean algebra. Let $X = B$. Define by induction the map $h : \{\text{formulas over } X\} \to B$ as follows: the basis is $h(x) = x$ for $x \in B$, and $h(\varphi \vee \psi) = h(\varphi) \vee h(\psi)$, etc. Let $\Gamma = \{\psi \mid h(\psi) = \mathbf{1}\}$. We claim that $\mathcal{F}_{X,\Gamma}$ and B are isomorphic under the map $H([\varphi]_\Gamma) \stackrel{\text{def}}{=} h(\varphi)$.

Note that Γ is closed under logical consequences: if $\Gamma \vDash \varphi$, then by compactness there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\{\psi_1, \ldots, \psi_n\} \vDash \varphi$, and $h(\psi_i) = \mathbf{1}$, $1 \le i \le n$, therefore $\mathbf{1} = h(\psi_1) \wedge \cdots \wedge h(\psi_n) \le h(\varphi)$, so $h(\varphi) = \mathbf{1}$, so $\varphi \in \Gamma$. Then H is well-defined and one-to-one, since $\varphi \equiv_\Gamma \psi$ iff $\Gamma \vDash \varphi \leftrightarrow \psi$ iff $h(\varphi \leftrightarrow \psi) = \mathbf{1}$ iff $h(\varphi) = h(\psi)$. It is surely onto, since every element of B is the image of a generator in X. Finally, H is easily shown to be a Boolean algebra homomorphism:

$$\begin{aligned} H([\varphi]_\Gamma \vee [\psi]_\Gamma) &= H([\varphi \vee \psi]_\Gamma) \\ &= h(\varphi \vee \psi) \\ &= h(\varphi) \vee h(\psi) \\ &= H([\varphi]_\Gamma) \vee H([\psi]_\Gamma), \end{aligned}$$

etc.