Logics of Programs	**Homework 2**
CS686 Spring 2001	**Solutions Posted Monday 2/26**

1. HKT Exercise 3.8 pp. 136–137. Use the equational axiomatization on p. 137. (In class I listed the De Morgan laws and the law of double negation as axioms, but they are actually redundant.)

2. HKT Exercise 3.11 p. 137.

3. (a) Let B be a Boolean algebra. Show that a set $F \subseteq B$ is a filter iff there exists a Boolean algebra A and a homomorphism $h : B \to A$ such that $F = \{x \mid h(x) = 1\}$.

   (b) Show that F is an ultrafilter iff there exists a homomorphism $h : B \to \mathbf{2}$ such that $F = \{x \mid h(x) = 1\}$, where $\mathbf{2}$ is the two-element Boolean algebra.

4. Let B be a set-theoretic Boolean algebra on a set X. Let Y be an arbitrary nonempty subset of X (not necessarily an element of B). Let B_Y be the set-theoretic Boolean algebra on Y whose elements are $\{x \cap Y \mid x \in B\}$. Show that the map $x \mapsto x \cap Y$ is a Boolean algebra homomorphism. What is its kernel?

5. An *atom* of a Boolean algebra is a minimal nonzero element. An element y is *atomless* if there is no atom $x \leq y$. A Boolean algebra is *atomic* if there are no atomless elements. Show:

   (a) Any Boolean algebra is a subalgebra of an atomic Boolean algebra. (*Hint.* Easy with Stone representation.)

   (b) Any two countable atomless Boolean algebras are isomorphic.

6. Let Γ be a consistent set of propositional formulas on a set of propositional letters X. Consider the Boolean algebra $\mathcal{F}_{X,\Gamma}$ of propositional formulas over X modulo the relation

$$\varphi \equiv_\Gamma \psi \quad \stackrel{\text{def}}{\iff} \quad \Gamma \vDash \varphi \leftrightarrow \psi;$$

   that is, $\varphi \equiv_\Gamma \psi$ if φ and ψ have the same truth value under all truth assignments to X that satisfy Γ. The elements of $\mathcal{F}_{X,\Gamma}$ are the $\equiv_\Gamma$-classes $[\varphi]_\Gamma = \{\psi \mid \psi \equiv_\Gamma \varphi\}$ with the natural definition of the Boolean operations: $[\varphi]_\Gamma \vee [\psi]_\Gamma \stackrel{\text{def}}{=} [\varphi \vee \psi]_\Gamma$, etc.

   (a) The *free Boolean algebra on generators* X is $\mathcal{F}_{X,\varnothing}$. Thus $\varphi \equiv_\varnothing \psi$ if $\varphi \leftrightarrow \psi$ is a propositional tautology. Show that if X has n elements, then $\mathcal{F}_X$ has 2^n atoms and 2^{2^n} elements.

   (b) Show that if X is infinite, then $\mathcal{F}_{X,\varnothing}$ is atomless.

(c) Describe a natural homomorphism $h : \mathcal{F}_{X,\varnothing} \to \mathcal{F}_{X,\Gamma}$. For this homomorphism, show that the following statements are equivalent:

(i) $[\neg\varphi]_\varnothing \in \ker h$ (the *kernel* of h = set of elements mapped to 0);

(ii) $[\varphi]_\varnothing$ is in the filter generated by the $\equiv_\varnothing$-classes of elements of Γ;

(iii) φ is a logical consequence of Γ (true in all truth assignments satisfying Γ).

(d) Show that every Boolean algebra is isomorphic to $\mathcal{F}_{X,\Gamma}$ for some X and Γ.