Logics of Programs	**Homework 1 Solutions**
CS686 Spring 2001	**Wednesday, January 31, 2001**

1. (a) More generally, the Cartesian product of finitely many well-founded ordered sets is well-founded in the componentwise product order. Let A be any nonempty set of pairs $(x, y) \in X \times Y$, where $(X, \leq)$ and $(Y, \leq)$ are well-founded ordered sets. The set $\{x \mid \exists y\ (x, y) \in A\}$ of all first components of pairs in A is a nonempty subset of the well-founded set X, so it has a minimal element x_0. The set $\{y \mid (x_0, y) \in A\}$ of all second components of pairs in A whose first component is x_0 is a nonempty subset of the well-founded set Y, so it has a minimal element y_0. The pair (x_0, y_0) is a minimal element of A, since no pair has a smaller first component by the choice of x_0 and no pair with first component x_0 has a smaller second component by the choice of y_0.

   Note that (x_0, y_0) is not necessarily a mini*mum* element of A. There could be an (x, y) with $x > x_0$ and $y < y_0$.

   (b) We show that for any ordinals α, β, either $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$ by induction on the well-founded relation of (a). For $\gamma \in \alpha$, by the induction hypothesis, either (i) $\gamma \in \beta$, (ii) $\gamma = \beta$, or (iii) $\beta \in \gamma$. If there exists $\gamma \in \alpha$ satisfying (ii) or (iii), then $\beta \in \alpha$ (directly in case (ii) and by transitivity in case (iii)), and we are done. Otherwise (i) holds for all $\gamma \in \alpha$, so $\alpha \subseteq \beta$. By a symmetric argument, either there exists $\delta \in \beta$ such that $\alpha = \delta$ or $\alpha \in \delta$, in which case $\alpha \in \beta$ we are done, or $\delta \in \alpha$ for all $\delta \in \beta$, in which case $\beta \subseteq \alpha$. Thus in the only remaining case, both $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$, so $\alpha = \beta$.

2. (a) Let the sets be A_i, $i < \omega$. Let $p : \omega^2 \to \omega$ be any one-to-one and onto pairing function, say $(i, j) \mapsto (i + j + 1)(i + j)/2 + j$. Map a to $p(i, j)$, where a is the j^{th} element of A_i and a is not in A_k for any $k < i$. This map puts $\bigcup_i A_i$ in one-to-one correspondence with a subset of ω, so it is countable.

   (b) The labels are successor ordinals by definition. For all $x \in T$, if $\mathrm{ord}_T(xn)$ is a countable ordinal for all $n \in \omega$ such that $xn \in T$, then $(\sup_{xn \in T} \mathrm{ord}_T(xn)) + 1$ is countable by (a), since by definition

   $$(\sup_{xn \in T} \mathrm{ord}_T(xn)) + 1 \;=\; \bigcup_{xn \in T} \mathrm{ord}_T(xn) \cup \{\bigcup_{xn \in T} \mathrm{ord}_T(xn)\},$$

   and this is a countable union of countable sets. By well-founded induction on T, any ordinal labeling any node of T is countable.

   Conversely, all countable successor ordinals can be so represented. Let $\alpha + 1$ be such an ordinal and suppose all successor ordinals $\beta < \alpha + 1$ are $\mathrm{ord}_T(\varepsilon)$ for some tree T. If α is a successor ordinal, let T be a tree such that $\alpha = \mathrm{ord}_T(\varepsilon)$. Build a new tree T' with a new root and a single child T. Formally, $T' = \{\varepsilon\} \cup \{0x \mid x \in T\}$. One can show by induction that $\mathrm{ord}_{T'}(0x) = \mathrm{ord}_T(x)$ for any x; in particular, $\mathrm{ord}_{T'}(0) = \mathrm{ord}_T(\varepsilon) = \alpha$, therefore $\mathrm{ord}_{T'}(\varepsilon) = \alpha + 1$. If α is a limit ordinal, let T_β be a tree such that $\beta = \mathrm{ord}_{T_\beta}(\varepsilon)$ for all successor ordinals $\beta < \alpha$. These exist by the induction hypothesis. Build a new tree T' with a new root and children T_β. This is possible because there are only countably many such β. Then $\mathrm{ord}_T(\varepsilon) = (\sup_{\beta<\alpha} \mathrm{ord}_{T_\beta}(\varepsilon)) + 1 = (\sup_{\beta<\alpha} \beta) + 1 = \alpha + 1$.

3. I have a very embarrassing erratum to report. There is a condition omitted from the definition of continuous set operator on p. 17 of HKT that makes part of this exercise unsolvable. The definition should read: A set operator τ is said to be continuous if for every *countable* chain of sets $\mathcal{C}$, ... This is the definition that we want, because it is all we need to ensure that the closure ordinal of τ is ω (Exercise 4). I'm not sure how the countability condition got omitted but it is definitely supposed to be there. Without it, it is impossible to construct a continuous operator that is not finitary. I sincerely regret the error and humbly apologize to those of you who wasted time on this exercise.

   Continuous operators are monotone: if τ is continuous,

$$\begin{aligned} A \subseteq B &\Leftrightarrow B = A \cup B \\ &\Rightarrow \tau(B) = \tau(A \cup B) \\ &\Leftrightarrow \tau(B) = \tau(A) \cup \tau(B) \quad \text{by continuity} \\ &\Leftrightarrow \tau(A) \subseteq \tau(B). \end{aligned}$$

   Finitary operators are continuous: if τ is finitary and $\mathcal{C}$ is a countable chain,

$$\begin{aligned} \tau(\bigcup \mathcal{C}) &= \bigcup\{\tau(B) \mid B \subseteq \bigcup \mathcal{C},\ B \text{ finite}\} \\ &= \bigcup\{\tau(B) \mid \exists C \in \mathcal{C}\ B \subseteq C,\ B \text{ finite}\} \\ &= \bigcup \bigcup_{C\in\mathcal{C}} \{\tau(B) \mid B \subseteq C,\ B \text{ finite}\} \\ &= \bigcup_{C\in\mathcal{C}} \bigcup\{\tau(B) \mid B \subseteq C,\ B \text{ finite}\} \\ &= \bigcup_{C\in\mathcal{C}} \tau(C). \end{aligned}$$

   The set operator on subsets of ω given by

$$A \ \mapsto \ \begin{cases} A, & \text{if } A \text{ is finite} \\ \omega, & \text{otherwise} \end{cases}$$

is monotone but not continuous. The set operator on subsets of ω_1 given by

$$A \;\mapsto\; \begin{cases} A, & \text{if } A \text{ is countable} \\ \omega_1, & \text{otherwise} \end{cases}$$

is continuous but not finitary. It is continuous because for any countable chain A_i, either some A_i is uncountable, in which case $\tau(\bigcup_i A_i) = \bigcup_i \tau(A_i) = \omega_1$, or not, in which case by 2a, $\bigcup_i A_i$ is countable and $\tau(\bigcup_i A_i) = \bigcup_i \tau(A_i) = \bigcup_i A_i$. It is not finitary, because for any uncountable set A, $\tau(A) = \omega_1$ but $\bigcup\{\tau(B) \mid B \subseteq A,\ B \text{ finite}\} = A$.

Without the condition of countability in the definition of continuous, continuous implies finitary. This can be shown by transfinite induction as follows. Recall $A \equiv B$ if there exists $f : A \xrightarrow[\text{onto}]{1-1} B$. The *cardinality* of A is the least ordinal α such that $\alpha \equiv A$. The cardinality of A is either finite or a limit ordinal, since $\alpha + 1 \equiv \alpha$ for infinite α (map $\beta \mapsto \beta$ for $\omega \le \beta < \alpha$, $n \mapsto n+1$ for $n < \omega$, and $\alpha \mapsto 0$).

Now suppose τ is a set operator on X and $\tau(\bigcup \mathcal{C}) = \bigcup_{C \in \mathcal{C}} \tau(C)$ for any chain $\mathcal{C}$, countable or not. For any $A \subseteq X$, let α be its cardinality and $f : \alpha \xrightarrow[\text{onto}]{1-1} A$. If A is finite, there is nothing to prove. Otherwise, α is a limit ordinal. For any $\beta < \alpha$, define $A_\beta = \{f(\gamma) \mid \gamma < \beta\}$. Then the A_β form a chain and $\bigcup_{\beta<\alpha} A_\beta = A$. Moreover, all the A_β are of smaller cardinality since $A_\beta \equiv \beta < \alpha$, so by the induction hypothesis, $\tau(A_\beta) = \bigcup\{\tau(B) \mid B \subseteq A_\beta,\ B \text{ finite}\}$. Then

$$\begin{aligned}
\tau(A) &= \tau(\bigcup_{\beta<\alpha} A_\beta) \\
&= \bigcup_{\beta<\alpha} \tau(A_\beta) \quad \text{by continuity} \\
&= \bigcup_{\beta<\alpha} \bigcup_{\substack{B \subseteq A_\beta \\ B \text{ finite}}} \tau(B) \\
&= \bigcup_{\substack{B \subseteq A \\ B \text{ finite}}} \tau(B).
\end{aligned}$$

The last equation follows from the fact that for finite B, $B \subseteq A$ iff $B \subseteq A_\beta$ for some $\beta < \alpha$.

4. If τ is continuous, then

$$\begin{aligned}
\tau^{\omega+1}(\varnothing) &= \tau(\tau^{\omega}(\varnothing)) \\
&= \tau(\bigcup_{n<\omega} \tau^{n}(\varnothing)) \\
&= \bigcup_{n<\omega} \tau(\tau^{n}(\varnothing)) \\
&= \varnothing \cup \bigcup_{n<\omega} \tau^{n+1}(\varnothing) \\
&= \bigcup_{n<\omega} \tau^{n}(\varnothing) \\
&= \tau^{\omega}(\varnothing).
\end{aligned}$$

For an example of a τ whose closure ordinal is $\omega + 1$, let X be the set of nodes of the tree on p. 51 of HKT, and let

$$\tau(A) = \{x \mid \text{all successors of } x \text{ are in } A\}.$$

Then $\tau(\varnothing)$ are the leaves, $\tau^{2}(\varnothing)$ are the leaves and the nodes above the leaves, and so on; $\tau^{\omega}(\varnothing)$ is the set of all nodes except the root, and $\tau^{\omega+1}(\varnothing)$ is the set of all nodes, and this is the least fixpoint.