Logics of Programs	**Homework 1 Solutions**
CS686 Spring 1998	**February 10, 1998**

1. Let A be an infinite subset of a compact space X. Suppose A had no accumulation point. Then for every $x \in X$, there would be an open set U_x such that $x \in U_x$ and $U_x \cap A$ is finite. But the union $\bigcup_{x \in X} U_x$ is an open cover of X, therefore has a finite subcover. Since A is infinite, one of the sets in the subcover must have an infinite intersection with A, contradiction.

   Alternative proof: Let $\mathcal{C}$ be the collection of all closed subsets of X whose intersection with A is cofinite (contains all but finitely many elements of A). The set $\mathcal{C}$ has the finite intersection property, therefore has a nonempty intersection. Claim: any $x \in \bigcap \mathcal{C}$ is an accumulation point of A. Any open set U containing x has an infinite intersection with A, otherwise its complement $\sim U$ would have been in $\mathcal{C}$ and $x \in \sim U$.

2. Let A be a closed subset of a compact space X. Let $\mathcal{C}$ be any open cover of A. The collection $\mathcal{C}$ along with the complement of A forms an open cover of the whole space X. By compactness, there is a finite subcover. Those elements of the subcover in $\mathcal{C}$ form a finite subcover of A.

3. Let $h : X \to Y$ be a continuous map. Let $\mathcal{C}$ be an open cover of $h(A)$, where $A \subseteq X$ is compact. Then $\{h^{-1}(U) \mid U \in \mathcal{C}\}$ is an open cover of A. By compactness, there is a finite subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\{h^{-1}(U) \mid U \in \mathcal{C}_0\}$ is a cover of A. Then $\mathcal{C}_0$ is a finite subcover of $h(A)$, since

$$\begin{aligned} h(A) &\subseteq h(\bigcup_{U \in \mathcal{C}_0} h^{-1}(U)) \\ &\subseteq h(h^{-1}(\bigcup_{U \in \mathcal{C}_0} U)) \\ &\subseteq (\text{not} = !!!) \bigcup_{U \in \mathcal{C}_0} U. \end{aligned}$$

4. Suppose $h : X \to Y$ is continuous, X is compact, and Y is Hausdorff. It suffices to show that every point in $Y - h(X)$ is contained in an open set disjoint from $h(X)$, because then $Y - h(X)$ will be the union of all these open sets, which is open, so $h(X)$ will be closed.

   Fix $y \in Y - h(X)$. Since Y is Hausdorff, for every $x \in X$ there exist disjoint open sets $U_x, V_x \subseteq Y$ such that $h(x) \in U_x$ and $y \in V_x$. The collection of all $h^{-1}(U_x)$ is an open cover of X, therefore has a finite subcover $\{h^{-1}(U_x) \mid x \in F\}$, where F is a finite set. By an argument in the previous exercise, $\bigcap_{x \in F} U_x$ is an open cover of $h(X)$, therefore $\bigcap_{x \in F} V_x$ is an open subset of Y disjoint from $h(X)$ containing y.

5. (a) It is the Boolean algebra of sets of truth valuations. A truth valuation corresponds to an ultrafilter: a propositional formula φ is in the ultrafilter corresponding to σ if $\sigma(\varphi) = \mathbf{1}$. There are 2^n truth valuations for X of cardinality n and 2^{2^n} subsets, so this BA has 2^{2^n} elements.

   (b) It is the Cantor space, the infinite binary tree $\{0,1\}^\omega$ with the product topology.

6. (a) The inverse image of any filter in A under a homomorphism $h : B \to A$ is a filter in B. In particular, the inverse image of the trivial filter $\{1\}$ is a filter in B. For example, if F is a filter in A, then

$$\begin{aligned} x, y \in h^{-1}(F) &\Rightarrow h(x), h(y) \in F \\ &\Rightarrow h(y) \wedge h(y) \in F \\ &\Rightarrow h(x \wedge y) \in F \\ &\Rightarrow x \wedge y \in h^{-1}(F). \end{aligned}$$

   The other properties of filters are argued similarly, using the facts that h is monotone with respect to the natural order in A and B, $h(0) = 0$, and $h(1) = 1$.

   Conversely, given any filter F, define $x \equiv y$ if $x \leftrightarrow y \in F$, where $x \leftrightarrow y \stackrel{\text{def}}{=} (x \wedge y) \vee (\neg x \wedge \neg y)$ (intuitively: two propositions φ and ψ are propositionally equivalent iff the proposition $\varphi \leftrightarrow \psi$ is a tautology). The relation $\equiv$ is a Boolean algebra congruence on B; that is, an equivalence relation that respects all the Boolean algebra operations in the sense that $x \equiv y \Rightarrow x \vee z \equiv y \vee z$, etc. This allows us to form the *quotient algebra* $B/{\equiv}$ whose elements are the congruence classes $[x] \stackrel{\text{def}}{=} \{y \mid x \equiv y\}$ with operations $[x] \vee [y] \stackrel{\text{def}}{=} [x \vee y]$, etc., and these operations are well-defined. The filter F is the inverse image of $\{1\}$ under the natural homomorphism $x \mapsto [x] : B \to B/{\equiv}$.

   (b) This follows immediately from part (a) and the property of ultrafilters that for all $x \in B$ and ultrafilters u, either $x \in u$ or $\neg x \in u$.