**CS 6840 Algorithmic Game Theory** April 17, 2020

# Lecture 27: Bayes extension of smoothness in auctions

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Today's topic: Auctions and Price of Anarchy, Bayes Version: an extension theorem. This lecture will be similar to the March 13th lecture, the last live lecture.

Next week, we will switch to learning, starting with auctions. Auctions will be the worst setting we look at; bounds for auctions are hard to prove.

## 1 Recap

An auction is $(\lambda, \mu)$-smooth if for all valuations $v_1, \ldots, v_k$ (these can be vectors) there is a special bid $b_i^*$ for all players $i$ such that for all bid vectors $b$,

$$\sum_i \mathbb{E}_{b_i^*}\left(u_i(b_i^*, b_{-i})\right) \geq \lambda \operatorname{Opt}(v) - \mu \operatorname{Rev}(b)$$

$b_i^*$ can be random, and $b^*$ can depend on $v$.

We have seen

- all-pay is $(\frac{1}{2}, 1)$-smooth
- multi-item, first price, unit demand $(\frac{1}{2}, 1)$-smooth

**Theorem**: If an auction is $(\lambda, \mu)$-smooth and $b$ is a pure or mixed Nash, then

$$\mathbb{E}_b(SW(b)) \geq \frac{\lambda}{\max(1, \mu)} \operatorname{Opt}$$

## 2 Extension to Bayes

If our auction is $(\lambda, \mu)$-smooth, $v_i \in \mathcal{F}_i$ are random and independent, and $b$ is a Bayes Nash, then

$$\mathbb{E}_{b,v}(SW(b, v)) \geq \frac{\lambda}{\max(1, \mu)} \mathbb{E}_v(\operatorname{Opt}(v))$$

The independence assumption is necessary to the proof, but perhaps not a very realistic assumption.

*Proof*: Because of smoothness, for every $v_1, \ldots, v_k$, we have a corresponding $b_i^*(v)$.
We used to have

$$\mathbb{E}_{v,b}(u_i(b)|v_i) \geq \mathbb{E}(u_i(b_i^*(v))|v_i);$$

however, we cannot consider this, since it depends on $v_{-i}$. This doesn't make sense, since you can't "magically" know other people's values. We need a new $b_i^*$.

To do this, we will take a random sample $\bar{v}_{-i}$ from the distributions $\mathcal{F}_{-i}$ and use $b_i^*(v_i, \bar{v}_{-i})$. Using this new $b_i^*$, we need to evaluate

$$\mathbb{E}_{v_i}\left[\mathbb{E}_{\bar{v}}\left(u_i(b_i^*(v_i, \bar{v}_{-i}), b_{-i}(v)\right)\right]$$

Note that $b_{-i}(v)$ has no dependence on $v_i$. Thus, we can rename $v_i$ to $\bar{v}_i$. This expression then becomes

$$\mathbb{E}_{\bar{v}}(u_i(b_i^*(\bar{v}), b_{-i}(v)))$$

Then, taking the expectation over $v_{-i}$, we get (noting that $v_i$ appears only in $\bar{v}$ in the following expression)

$$\mathbb{E}_v\left(\mathbb{E}_{\bar{v}}(u_i(b_i^*(\bar{v}), b_{-i}(v)))\right)$$

Using the above, we can show $(\lambda, \mu)$-smoothness:

$$\begin{aligned}
\mathbb{E}_v\left(\sum_i \mathbb{E}_{\bar{v}}(u_i(b_i^*(v_i, \bar{v}_{-i}), b(v)))\right) &= \sum_i \mathbb{E}_v\mathbb{E}_{\bar{v}}(u_i(b_i^*(v_i, \bar{v}_{-i}), b(v))) && \text{(linearity)}\\
&= \sum_i \mathbb{E}_v\mathbb{E}_{\bar{v}}(u_i(b_i^*(\bar{v}_i), b(v))) && \text{(previously explained renaming)}\\
&= \mathbb{E}_{\bar{v},v}\left(\sum_i u_i(b_i^*(\bar{v}_i), b(v))\right) && \text{(linearity)}\\
&\geq \mathbb{E}_{\bar{v},v}(\lambda\,\mathrm{Opt}(\bar{v}) - \mu\,\mathrm{Rev}(b(v))) && \text{(smoothness)}\\
&= \lambda\mathbb{E}_{\bar{v}}(\mathrm{Opt}(\bar{v})) - \mu\mathbb{E}_v(\mathrm{Rev}(b(v))) \\
&= \lambda\mathbb{E}_v(\mathrm{Opt}(v)) - \mu\mathbb{E}_v(\mathrm{Rev}(b(v))) && (\bar{v} \text{ and } v \text{ have the same distribution})
\end{aligned}$$

This implies that we are $(\lambda, \mu)$-smooth in expectation.

# 3 Price of Anarchy

Now, we want to prove the Price of Anarchy bound.

$$\begin{aligned}
\mathbb{E}_{v,b}\sum_i(u_i(b,v)) &\geq \mathbb{E}_{v,b,\bar{v}}\left[\sum\left[u_i(b_i^*(v_i,\bar{v}_{-i}), b_{-i})\right]\right]\\
&\geq \lambda\mathbb{E}_i(\mathrm{Opt}(v)) - \mu\mathbb{E}_v(\mathrm{Rev}(b(v))) && \text{(from above)}
\end{aligned}$$

We can rearrange this to get

$$\begin{aligned}
\mathbb{E}_{v,b}(SW(b,v)) &= \mathbb{E}_{b,v}\left(\sum u_i(b,v)\right) + \mathbb{E}_{b,v}(\mathrm{Rev}(b))\\
&\geq \frac{\lambda}{\max(1,\mu)}\mathbb{E}_v(\mathrm{Opt}(v))
\end{aligned}$$

Note: It was important that we could apply smoothness with any $b$, and we will continue to use this in the future.