CS 6840 Algorithmic Game Theory February 12, 2020

Lecture 9: Smoothness Proofs and the Hotelling Game

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In the last two lectures we saw smoothness technique to bound the price of anarchy for routing games with both atomic and continuous flow. To do so, we used a magic inequality involving the delay function. It is this property of the cost function that we generalize for smoothness proofs. In this lecture, we see smoothness technique as a recipe for bounding the price of anarchy for general games and illustrate the technique on a specific example called the hotelling game.

General Setting for the Smoothness Proofs

We have a game with n players and each player has a set of *strategies*. Let the i^{th} player has a set of strategies S_i. In the game, the player i chooses a strategy $s_i \in S_i$, and define $\mathbf{s} = (s_1, \ldots, s_n)$. The player i incurs a cost $c_i(\mathbf{s})$, where c_i is the cost function for the i^{th} player. We define the social cost $SC(\mathbf{s}) = \sum_{i=1}^{n} c_i(\mathbf{s})$.
We also have that if $\mathbf{s}$ is the set of chosen strategies at Nash Equilibrium then:
$\forall i\ \forall s'_i \in S_i,\ c_i(\mathbf{s}) \leq c_i(s'_i, \mathbf{s_{-i}})$
and conversely, the above condition implies $\mathbf{s}$ is at Nash Equilibrium, where $s'_i, \mathbf{s_{-i}}$ represents s_i replaced by s'_i in $\mathbf{s}$.
The above means that Nash occurs if and only if every player has chosen the best strategy for themselves keeping others' strategies fixed.
For example in the Braess Paradox network with atomic routing for 100 players with $d_e(x) = x$ and constant delay of 100, the LHS of above is 200, and the RHS is also 200 for the other two routes which are not the Nash strategy.

Smoothness Proof for PoA

Let $\mathbf{s}^* = (s_i^*, \ldots, s_n^*)$ be a vector which minimizes $SC(\mathbf{s}^*)$.
We add an additional assumption that such a minimum exists. For example, we can have each S_i to be finite to guarantee this.
Suppose $\mathbf{s}$ is at Nash Equilibrium, then by the Nash condition above we have:
$\forall i,\ c_i(\mathbf{s}) \leq c_i(s_i^*, \mathbf{s_{-i}})$
If we add the inequalities above for all i, then we have
$SC(\mathbf{s}) = \sum_{i=1}^{n} c_i(\mathbf{s}) \leq \sum_{i=1}^{n} c_i(s_i^*, \mathbf{s_{-i}})$. Now we use a special assumption called the *smoothness* property (generalization of the magic inequality) of the cost functions that:

$$\forall i\ \forall \mathbf{s}, \mathbf{s}^*, \sum_{i=1}^{n} c_i(s_i^*, \mathbf{s_{-i}}) \leq \lambda SC(\mathbf{s}^*) + \mu SC(\mathbf{s})\ \textit{for some}\ \lambda, \mu \geq 0\ \textit{and}\ \mu < 1$$

Thus, we have:

$$SC(\mathbf{s}) = \sum_{i=1}^{n} c_i(\mathbf{s}) \leq \sum_{i=1}^{n} c_i(s_i^*, \mathbf{s_{-i}}) \leq \lambda SC(\mathbf{s}^*) + \mu SC(\mathbf{s})$$

rearranging we get

$$\frac{SC(\mathbf{s})}{SC(\mathbf{s}^*)} \leq \frac{\lambda}{1-\mu}$$

So, we have the following theorem that:

Theorem 1. *If the cost functions are (λ, μ) smooth and the social cost has a minimum and Nash Equilibrium also exists, then the price of anarchy of such a game is bounded by*

$$PoA \leq \frac{\lambda}{1-\mu}$$

Smoothness Bound for another Variant: Value Maximization

In this model again we have n players with their own sets of strategies as before, but instead of the cost function we have value function $v_i(\mathbf{s})$ for the i^{th} player. Each player intends to maximize their value, so at Nash Equilibrium $\mathbf{s}$, we have:

$$v_i(\mathbf{s}) \geq v_i(s'_i, \mathbf{s_{-i}}) \quad \forall i, \ \forall s'_i \in S_i$$

Also instead of the social cost, we have *social welfare* defined by

$$SW(\mathbf{s}) = \sum_{i=1}^{n} v_i(\mathbf{s})$$

and the price of anarchy is defined as the ratio of maximum social welfare to social welfare of the Nash with minimum social welfare.
As before we assume that Nash Equilibrium and maximum of social welfare both exist.
Here, the value functions are (λ, μ) smooth if they satisfy the following inequality $\forall \mathbf{s}, \mathbf{s}^*$:

$$\sum_{i=1}^{n} v_i(\mathbf{s_i^*}, \mathbf{s_{-i}}) \geq \lambda SW(\mathbf{s}^*) - \mu SW(\mathbf{s})$$

for some $\lambda > 0$, and $\mu \geq 0$. Arguing as before, using Nash condition with $\mathbf{s_i^*}$ for S'_i and adding up for each player, we get:

$$SW(\mathbf{s}) \geq \sum_{i=1}^{n} v_i(\mathbf{s_i^*}, \mathbf{s_{-i}}) \geq \lambda SW(\mathbf{s}^*) - \mu SW(\mathbf{s})$$

where the last inequality uses the smoothness condition. Now, the above holds for any Nash $\mathbf{s}$, so we choose $\mathbf{s}$ to be the Nash with the minimum social welfare and $\mathbf{s}^*$ to be a set of chosen strategies that maximizes social welfare, then by rearranging the above we have the following theorem:

Theorem 2. *If the cost functions are (λ, μ) smooth and the social welfare has a maximum and Nash Equilibrium also exists, then the price of anarchy of such a game is bounded by*

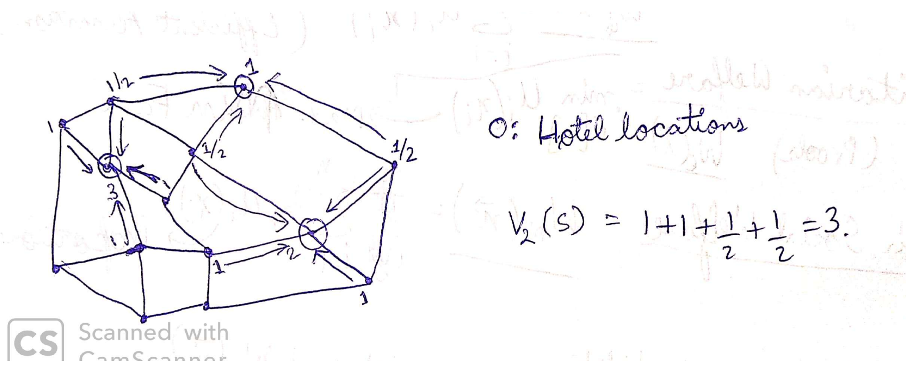
$$PoA = \frac{SW(\mathbf{s}^*)}{SW(\mathbf{s})} \leq \frac{1+\mu}{\lambda}$$

Hotelling Game: A smooth value maximization game

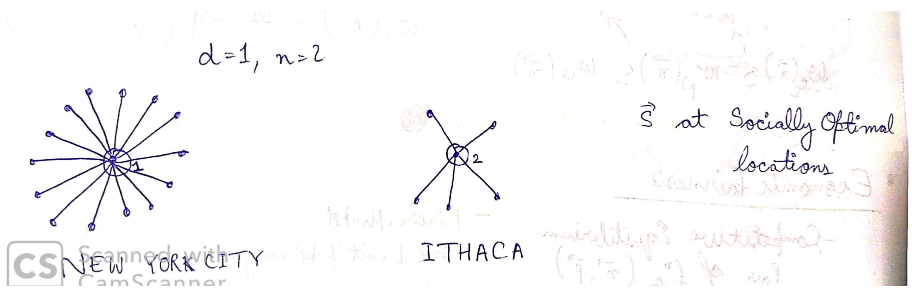
Consider an undirected graph $G = (V, E)$ and a non-negative integer d (as a parameter for the game) and n players. The rules of the hotelling game are:

- Each player chooses a vertex (need not be unique) in the graph and sets up a hotel.
- Each vertex has a customer who chooses to go to the nearest hotel among the hotels within a distance of d from its location.
- The value for player i is given by $v_i(\mathbf{s})$ = number of customers who choose s_i, the i^{th} player's hotel, where $\mathbf{s} = (s_1, \ldots, s_n)$ $s_i \in V$ is the location of player i's hotel.
- In case of a tie, i.e. if two or more hotels are at the same nearest distance ($\leq d$) from a customer, the customers value of 1 is evenly distributed among the contender hotels.

For example, in the example below for $d = 1, n = 3$: $v_2 = 1 + 1 + \frac{1}{2} + \frac{1}{2} = 3$.



Next we see if Nash is socially optimal? The answer is no as illustrated by the example below. Here, we have $d = 1$, $n = 2$, and the Nash would be for both players to set up their hotel at New York City but the social optimal would be to for one player to setup a hotel at New York City and the other at Ithaca. Next we bound the price of anarchy for this game.



Lemma 1. *The hotelling game is* $(1, 1)$ *smooth.*

Proof. *Next Lecture.* ■

Theorem 3. *The price of anarchy for the hotelling game is at most 2.*

Proof. Using the lemma above and the previous theorem on price of anarchy bounds for smooth value maximization games:

$$PoA \leq \frac{1 + \mu}{\lambda} = 2$$

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