CS 6840 Algorithmic Game Theory March 6, 2020

Lecture 18: Cooperation and Strong Nash

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In today's lecture, we will discuss collaboration in games. We will focus on the cost-sharing game: a congestion game where player's cost of an element e shared by k players is c_e/k. Recall the bad example: with n players and 2 options from everyone: A and B with $c_A = n$ and $c_B = 1 + \epsilon$. There are two Nash equilibria: all choosing A for a total cost of n or all choosing B for a total ost of $1 + \epsilon$.

The question we raise today is: if the players can collaborate and move together: wouldn't they want to switch to the cheaper solution. What do we mean by "cooperate"? One notion is that of *Strong Nash*. A strategy vector $s = (s_1, \ldots, s_n)$ is a strong Nash equilibrium (for a game with costs) if for all subset of players A, and all alternate strategy vectors s'_A for this subset we cannot have that

$$c_i(s) \geq c_i(s'_A, s_{-A}) \qquad \forall i \in A$$

with at least one of these inequalities holding strictly. We use $c_i(s'_A, s_{-A})$ to denote the cost of player i when players in A to strategy s'_i and players outside A to s_i. The idea is that if there is such a subset, than subset A can jointly deviate to the new solution. A strong Nash is a a solution, when no such subset exists. (Note that this solution doesn't allow side payments: a player i cannot compensate a different player for a worse solution by paying the on the side.)

Observe that a strong Nash is a Nash equilibrium, as A cane be a set of just a single element, and then the condition is just the same as Nash. Also observe that in the simple example above, the good solution of cost $1 + \epsilon$ is strong Nash, but the bad solution of cost n is not.

We will discuss two questions.

- Does a strong Nash guaranteed to exists?
- Can we say something good about the quality of a strong Nash

Existence of strong Nash

Consider a two payer cost sharing in the network below

Note that there is a unique Nash equilibrium in this game of cost 22 marked on the picture. And this is not a strong Nash, if both players together deviate to the optimal solution both of their cost decreases. So the game has no strong Nash! In fact, note that this is the same game as the classical prisoner's dilemma:

	C	D
C	$10, 10$	$8, 13$
D	$13, 8$	$11, 11$

where C=cooperate corresponding of them playing their part in the optimum, and D=defect corresponds to them playing their part in the Nash.

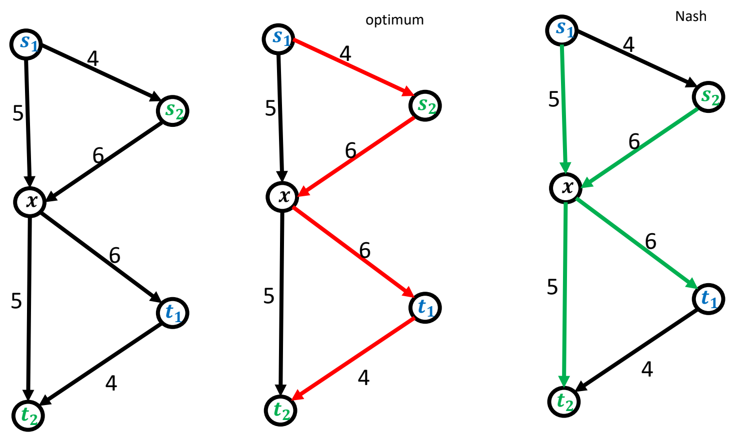

Figure 1: cost in a 2-player cost-sharing. Optimal solution in red of cost 20, and unique Nash equilibrium of cost 22.

Quality of strong Nash

The good news is that if there is a strong Nash, it has good quality. Before we state the theorem, want to think about a different example. Recall the bad example for best Nash from lecture on February 3rd. This example had n players, and the unique Nash equilibrium was a factor of $H_n = \sum_{i=1}^{n} \frac{1}{i}$ more expensive than the optimum.

Claim: the unique Nash in this example strong Nash.

Proof: Consider any subset A of players. Let $i \in A$ be the highest indexed player. So $i \geq |A|$. in the Nash solution i pays $1/i$. In contrast if the set A deviates to the optimum, each will pay $\frac{1+\epsilon}{|A|}$ which is more, so this is not a viable deviation.

Next class we will prove that this is the worst case, that is: if the game has a strong Nash equilibrium, than the cost of that equilibrium is at most an H_n factor higher than the optimal cost.

We note that for Nash equilibrium version of the game we have some result outcomes of game dynamics. We know that if a game has good price of anarchy via a smoothness proof than learning dynamic will get a solution not much worse (loosing a bit due to the small regret). It is open if something similar may be true for dynamics that allows cooperative moves.