CS 6840 Algorithmic Game Theory April 8, 2020

Lecture 23: All-pay auction analysis

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Anouncements

- The project proposal is due Friday: it should be 1-1.5 pages. It's okay to be rough, but it should include some sort of direction about where you're headed.
- Problem set 3 will be released Friday or over the weekend.
- Still planning on going with the plan for the rest of the semester being 2 homeworks and the project, with the project deadline pushed back to exam period.

Exact analysis of all-pay auction with two players

As a reminder, in the all-pay auction, losers and winners all pay their bid. We've previously discussed how there is no pure-strategy Nash, but it turns out that there are Nash solutions!

Example 1. *$n = 2$ bidders, with $v_1 = v_2 = 1$. We claim that bidding uniformly at random, i.e., $b_1, b_2 \sim [0, 1]$ is Nash. (It turns out that this is also the unique Nash solution, which we won't prove here).*

Proof. *Suppose $b_1 \sim [0, 1]$. If player 2 bids $x \in [0, 1]$, what is his expected value? Because it's an all-pay auction, she has to pay x regardless, and gets $+1$ if she wins. What is the probability of winning?*

$$\Pr(\textit{winning}) = \Pr(b_1 < x) = x$$

As the event $b_1 = x$ has zero measure, we do not need to worry about ties. Therefore the expected value of any bid x is $x - x = 0$ always.

Let's consider another example, but here allow the two players to have different values.

Example 2. *Here, let's set the values so $v_1 = 1, v_2 = 2$. We claim that the following is a Nash equilibrium:*

$$b_2 \sim [0, 1] \textit{ uniformly}$$

and

$$b_1 = \begin{cases} b_1 = 0 & w.p.\ \frac{1}{2} \\ b_1 \sim [0, 1] \textit{ unif} & w.p.\ \frac{1}{2} \end{cases}$$

Proof *For this proof, we need to show that, for all $i = 1, 2$, a player opposing i with bid x gives the same value for all x.*

<u>For bidder 1</u>: *Let's consider biddere 1 giving an arbitrary bid x. What is bidder 1's uility? We analyzed this case before: it's the same as in Example 1, so bidder 1 gets utility 0 no matter what they do.*

<u>For bidder 2</u>: *Let's consider an arbitrary bid from bidder 2 as x, when bidder 1 is drawing from the distribution given above. (We will assume $x \leq 1$, because bidder 2 bidding more than 1 doesn't make sense.) With probability 1/2, bidder 1 bids 0, and so bidder 2 wins (utility $2 - x$). With the remaining probability (1/2), the same analysis as before: probability of winning proportional to x.*

$$\Pr(\textit{bidder 2 wins with bid } x) = \underbrace{\frac{1}{2}}_{\textit{if } b_1 = 0} + \underbrace{\frac{x}{2}}_{\textit{if } b_1 > 0}$$

So the overall value to bidder 2 is

$$2\Pr(\textit{bidder 2 wins with bid } x) - x = 2\left(\frac{1}{2} + \frac{x}{2}\right) - x = 1$$

again independent of x.

Note: *These results tell us that the higher value bidder may not win! The case where the lower bidder wines happens with* $\frac{1}{4}$ *probability: there is a* $\frac{1}{2}$ *change that* $b_1 \neq 0$, *in which case each bidder winning is equally likely.*

Using smoothness inequalities

It turns out that working out the actual equilibrium is hard:

- It is hard to find! In this case, we only verified the answers as Nash, but to find them requires tricky calculus that gets more tricky the more players with different values we have.
- It requires knowledge of other people's values, which we might not have.

By assuming smoothness, we can import results from previously in the semester. Today, we'll focus on a single-item auctions, 1^{st} price and all-pay (because these are the only real auctions with strategy).

What is social welfare? You should feel free to push back on this, but here we will use the definition of social welfare as the sum of the values of people with items they won — in our case, it is the value of the winner (we only have one). This is equal to the sum of the utility of winner + the money to auctioneer.

Like previously, let's assume some special bid b_i^* for bidder i. We assume a (λ, μ) smoothness criterion like the following:

$$\sum_i u_i(b_i^*, b_{-i}) \geq \lambda \text{ Opt} - \underbrace{\mu \text{ Rev}(b)}_{\text{A portion of SW}(b)}$$

where the terms are:

- the first term is the utility of player i when they are bidding b_i^* and everyone else is bidding according to $\vec{b}$.
- Opt is the maximum possible social welfare
- $\text{Rev}(b)$ is the revenue of the auctioneer with bids b

Note: when we previously did smoothness analysis, the last term was the SW(b), the social welfare. Here, our social welfare term includes Rev(b), but the math works out in this case that we can simplify it for this case.

Theorem 1. *If an auction* (λ, μ) *smooth (as defined above), and* b *is a Nash equilibrium,*

$$\text{SW}(b) \geq \frac{\lambda}{\max(1, \mu)} \; Opt$$

Proof. If b is a Nash equilibrium,

$$u_i(b) \geq u_i(b_i^*, b_{-i})$$

We could sum over i on each side to obtain:

$$\sum_i u_i(b) \geq \sum_i u_i(b_i^*, b_{-i})$$

Then, we can add an inequality on the right from smoothness:

$$\sum_i u_i(b) \geq \sum_i u_i(b_i^*, b_{-i}) \geq \lambda \text{Opt} - \mu \text{Rev}(b)$$

By rewriting, we can come up with:

$$\sum_i u_i(b) + \mu \ \text{Rev}(b) \geq \lambda \text{Opt}$$

Using the definition of social welfare tells us that:

$$\max(1, \mu) \ \text{SW}(b) \geq \sum_i u_i(b) + \mu \ \text{Rev}(b) \geq \lambda \text{Opt}$$

as desired.

Note. We might want b_i or b_i^* to be random. It turns out that the natural extension works in expectation. If the below is satisfied:

$$\sum_i \mathbb{E}\Big[u_i(b_i^*, b_{-i})\Big] \geq \lambda \ \text{Opt} - \mu \ \mathbb{E}\Big[\text{Rev}(b)\Big]$$

then the same is true in expectation:

$$\sum_i \mathbb{E}[u_i(b)] \geq \sum_i \mathbb{E}[u_i(b_i^*, b_{-i})] \geq \lambda \ \text{Opt} - \mu \ \mathbb{E}\Big[\text{Rev}(b)\Big]$$

where the first inequality is by the statement that b is Nash, so

$$\sum_i \mathbb{E}[u_i(b)] + \mu \ \mathbb{E}\Big[\text{Rev}(b)\Big] \geq \lambda \text{Opt}$$

In the future we will look at Bayesian games where there is randomness over the values as well.

Summary

Today, we analyzed some equilibria of auctions, and we also used the much simpler smoothness framework with the usual “magic inequality” to get bounds. Next time, we will use more examples of the smoothness inequality and look at how we can prove it.