CS 6840: Algorithmic Game Theory	**Fall 2019**

Lecture 3: January 27

Lecturer: Éva Tardos *Scribe: Sloan Nietert*

3.1 Equilibrium Flows and Social Cost Optimization for Selfish Routing

Recall the set-up for a selfish routing problem on a directed network $G = (V, E)$. We are given a collection of distinct source-sink pairs (s_i, t_i) with associated demands r_i, along with edge-specific non-negative delay functions $d_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. A valid flow on this network is an assignment of non-negative values $f_P \geq 0$ to the paths of G such that

$$\sum_{P: s_i \to t_i} f_P = r_i \quad \text{for all } i,$$

summing over all $s_i \to t_i$ paths P. Further, we define the flow over an edge e to be

$$f(e) = \sum_{P \ni e} f_p$$

and the delay through a path P to be

$$\sum_{e \in P} d_e(f(e)).$$

We say that a flow f is a (Nash) equilibrium flow if $f_P > 0$ for an $s_i \to t_i$ path P only when it is a minimum delay $s_i \to t_i$ path. That is, for each other $s_i \to t_i$ path Q, we have

$$\sum_{e \in P} d_e(f(e)) \leq \sum_{e \in Q} d_e(f(e)).$$

Finally, we define the social cost of a flow f as

$$SC(f) = \sum_e f(e) d_e(f(e)),$$

i.e. the total delay experienced by the routed agents. For a moment, we relax our definition of social cost to

$$SC(f) = \sum_e c_e(f(e)),$$

allowing edge-specific costs $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which are continuous, differentiable, and monotonically increasing. Let's examine this function from an optimization perspective.

Proposition 3.1 *If a flow f minimizes social cost as defined above, then, for each path P with $f_P > 0$, we have*

$$\sum_{e \in P} c'_e(f(e)) \leq \sum_{e \in Q} c'_e(f(e))$$

for all paths Q between the same $s_i \to t_i$ pair.

Proof: Suppose for the sake of contradiction that $f_P > 0$ and

$$\sum_{e \in P} c'_e(f(e)) > \sum_{e \in Q} c'_e(f(e))$$

for such a path Q. Then, we would be able to move a small amount of traffic $\delta > 0$ from P to Q and decrease the social cost. Indeed, for each $e \in Q \setminus P$,

$$c_e(f(e) + \delta) = c_e(f(e)) + \delta c'_e(f(e)) + o(\delta)$$

and, for each $e \in P \setminus Q$,

$$c_e(f(e) - \delta) = c_e(f(e)) - \delta c'_e(f(e)) + o(\delta),$$

as $\delta \to 0$ [1].Thus, our adjustment to f would result in the following negative change in social cost,

$$\delta \left(\sum_{e \in Q \setminus P} c'_e(f(e)) - \sum_{e \in P \setminus Q} c'_e(f(e) \right) + o(\delta)$$

$$= \delta \left(\sum_{e \in Q} c'_e(f(e)) - \sum_{e \in P} c'_e(f(e) \right) + o(\delta) < 0,$$

for sufficiently small $\delta > 0$. We have contradicted our initial hypothesis, proving the proposition. ■

Claim 3.2 *The converse is true, assuming that each c_e is convex.*

Proof: In this case, we are dealing with a convex program, since SC is convex as a sum of convex functions and our constraints are linear. Thus, any local minimum of SC is a global minimum. It turns out that the relevant "local moves" to assess local optimality are of the previous form, routing an infinitesimal amount of flow from a path with positive flow to another between the same $s_i \to t_i$ pair[2]. By the same logic as before, all moves of this form will increase social cost, so flows satisfying our condition are indeed local (and thus global) minimizers of social cost. ■

Now, let's examine this convexity assumption with the standard costs $c_e(x) = xd_e(x)$. Differentiating, we find that

$$c'_e(x) = d_e(x) + xd'_e(x),$$

which intuitively makes sense, as $d_e(x)$ is a "selfish" term, quantifying the delay experienced by an individual if they switch to a path using edge e, and $xd'_e(x)$ multiplies the current traffic times the extra delay per unit traffic to get an "externality" term quantifying the additional delay experienced by others. A real function is convex if and only if its derivative is monotonically increasing, so our convexity assumption is rather mild, only requiring that $d'_e(x)$ not decrease so fast as to counteract multiplication by x. Under this assumption, we have a nice corollary.

Corollary 3.3 *If each c_e is convex, then a global minimizer of social cost can be computed in polynomial time.*

Since we are minimizing a convex objective function over a convex set, this problem is indeed computationally tractable[3].

[1] Recall that $g(x) = o(x)$ as $x \to 0$ if $g(x)/x \to 0$ as $x \to 0$. We can further assume that the remaining terms are $O(\delta^2)$ with some mild assumptions, but these aren't necessary.

[2] Alternatively, one can derive the condition of Proposition 3.1 from the Karush–Kuhn–Tucker (KKT) conditions.

[3] To be a bit more precise, we should make a change of variables and consider the flow at each edge associated with each $s_i \to t_i$ pair (to avoid dealing with exponentially many variables). Then, this problem can be tackled (up to some approximation error) in polynomial time using, say, the ellipsoid method.

Corollary 3.4 *If each d_e is continuous and monotonically increasing, then a Nash equilibrium for the selfish routing problem can be computed in polynomial time.*

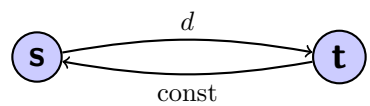
Proof: We can reduce this to the previous case by introducing costs

$$\overline{c}_e(x) = \int_0^x d_e(y)dy.$$

These new costs are convex, since each d_e is monotonically increasing. Thus, our new objective $\sum_e \overline{c}_e(f(e))$ is also convex, and we can find a minimizing flow in polynomial time. Its optimizer must satisfy the condition of Proposition 3.1, which, by design, is identical to that for Nash equilibrium. ■

Next class, we will work towards the following theorem.

Theorem 3.5 *Let $\mathcal{D}$ be a class of possible delay functions containing all non-negative constants, such that each $d \in \mathcal{D}$ is monotonically increasing, continuous, and continuously differentiable, with $x \mapsto xd(x)$ convex. Then for any input to the routing problem using delay functions from $\mathcal{D}$ (i.e. any graph G, input-output pairs s_i, t_i, demands r_i, and assignments $e \mapsto d_e \in \mathcal{D}$), the price of anarchy (PoA) for this instance is no greater than that for the following two edge graph G_0,*



where the constant and delay function d are chosen from $\mathcal{D}$ to maximize PoA. That is,

$$\max_{f \text{ Nash}, f^* \text{on } G} \frac{SC_G(f)}{SC_G(f^*)} \leq \max_{f \text{ Nash}, f^* \text{on } G_0} \frac{SC_{G_0}(f)}{SC_{G_0}(f^*)},$$

(identifying the graphs with their instances).