CS 6840 Algorithmic Game Theory April 15, 2020

Lecture 26: Smoothness for all-pay auction

Instructor: Eva Tardos *Scribe: Sotiris Ntanavaras, Owen Sconzo*

In today's lecture we focus on all-pay auctions with a single item, where players bid on a single items and both winners and losers pay their bids, while only their winner gets their value. Recall that such no pure Nash exists for such auctions, as shown in lecture 22, where in lecture 23 we analyzed a special case of this auction, which had $n = 2$ bidders, and based on their values we get the following mixed Nash:

- In the case where $v_1 = v_2 = 1$, both players bidding in $[0, 1]$ uniformly at random is a Nash.
- In the case where $v_1 = 1, v_2 = 2$, player 2 bidding uniformly in $[0, 1]$, and player 1 bidding either 0 or uniformly in $[0, 1]$, each with probability $1/2$ is a Nash.

Price of Anarchy for single item all-pay auction

Formally in an all-pay single item auction, we have n players, and each one of them has some value v_i for the items and bids some amount b_i. The player with the highest bid wins the item and has utility $v_i - b_i$, while the rest of the players have utility $-b_i$. We focus on showing the following theorem for mixed Nash(since pure Nash does not exist):

Theorem 1. *Let b be a mixed Nash bidding strategy. Then*

$$\mathbb{E}[SW(b)] \geq \frac{1}{2} \max_i v_i$$

To prove the theorem we will use the following claim:

Claim 1. *All-pay single item auction is $\left(\frac{1}{2}, 1\right)$ smooth.*

Proof. Given that b is mixed Nash, for some player i and any bid b_i^*:

$$\mathbb{E}_b[u_i(b)] \geq \mathbb{E}_{b,b_i^*}[u_i(b_i^*, b_{-i})]$$

It is worth noting that the expectation on the RHS is taken over both b and b_i^*, meaning that the bid b^* may also be randomized. We are interested in a specific bid b_i^*, defined as follows:

- $b_i^* = 0$, if i does not have the max value of v_i. Then, note that $\mathbb{E}_b[u_i(b_i^*, b_{-i})] \geq -b_i^* = 0$.
- b_i^* selected uniformly at random in $[0, v_i]$, if i has the max value v_i. Then,

  $$\mathbb{E}_{b,b_i^*}[u_i(b_i^*, b_{-i})] = v_i \Pr(i \text{ wins}) - \frac{1}{2} v_i$$

  We can see that it is possible for the RHS to take negative values.

  To calculate what $\Pr(i \text{ wins})$ is equal to, note that i wins when $b_i^* > \max_{j \neq i} b_j = \bar{b}$.

  For a fixed b,

  $$\mathbb{E}_{b_i^*}[u_i(b_i^*, b_{-i})] = v_i \Pr(\text{i wins}) - \frac{1}{2} v_i = v_i \left(\frac{v_i - \bar{b}}{v_i} \right) - \frac{v_i}{2} = \frac{v_i}{2} - \bar{b} \geq \frac{v_i}{2} - Rev(b)$$

The second equality is because b_i^* is selected uniformly in $[0, v_i]$ and the inequality is due to the fact that in an all-pay auction, everyone pays their bid so the revenue is at least as much as the max bid.

Summing over all i, we get

$$\sum_i u_i(b_i^*, b_{-i}) \geq 0 + \frac{1}{2} \max_i v_i - Rev(b)$$

prooving the claim. ■

This has the following implications:

1. All pure Nash equalibria has $SW \geq \frac{1}{2}OPT$. To see this if b is a pure Nash, we have $u_i(b) \geq \mathbb{E}b_i^*[u_i(b_i^*, b_{-i})]$. Summing over all i yields $\sum_i u_i(b) \geq \frac{1}{2} \max_i v_i - Rev(b)$, and we get $SW(b) = Rev(b) = \sum_i u_i(b) \geq \frac{1}{2} \max_i v_i$.

   However, since we started the lecture by noting that no pure Nash exists for this setting, this observation is not very useful.

2. The same holds in expectation for mixed Nash, which is exactly Theorem 1. To see this, we have $\mathbb{E}_b[u_i(b)] \geq \mathbb{E}_{b,b_i^*}[u_i(b_i^*, b_{-i})]$ because of the Nash property, and summing over all i's

$$\begin{aligned}
\sum_i \mathbb{E}_b[u_i(b)] &\geq \sum_i \mathbb{E}_{b,b_i^*}[u_i(b_i^*, b_{-i})] \\
&= \mathbb{E}_b \left[\sum_i \mathbb{E}_{b_i^*}[u_i(b_i^*, b_{-i})] \right] \\
&\geq \mathbb{E}_b \left[\frac{1}{2} \max_i v_i - Rev(b) \right] \\
&= \frac{1}{2} \max_i v_i - \mathbb{E}_b[Rev(b)]
\end{aligned}$$

The first equality is due to linearity of expectation, the second inequality due to Claim 1. This implies $\mathbb{E}_b[SW(b)] = \mathbb{E}_b[Rev(b)] + \sum_i \mathbb{E}_b[u_i(b)] \geq \frac{1}{2} \max_i v_i$, which is what Theorem 1 states.

Note that all the analysis up to this point operates under the assumption of fixed v_i's. This allowed us to easily differentiate the strategy of player i based on whether or not i had the max value for the single item. Switching to a Bayesian version of the all-pay single item auction, we now have random v_i's for all players. So, without fixed v_i's, player i cannot know whether to select b_i^* such that $b_i^* = 0$ or $b_i^* \in U(0, v_i)$.

From this we can conclude that in all-pay single item auctions we can show a PoA bound of $E[SW(b)] \geq \frac{1}{2} max_i\ v_i$ for pure strategy and mixed strategy Nash equilibria, but not for the Bayesian version of this auction.