CS 6840 Algorithmic Game Theory April 29, 2020

Lecture 33: The VCG Mechanism & GSP

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Recap: Better Auction Format

Previously, we defined second-price single-item auctions. In such an auction, there are n players/bidders, each holding value v_i for the item, and placing a bid of b_i for the item. The winner of the auction is bidder that places the highest bid ($\text{argmax}_i b_i$), and pays a price of the second highest bid ($\max_{j \neq i} b_j$).

Recall that second-price single-item auctions are truthful. That is, for each player, bidding their perceived value of the item ($b_i = v_i$) is a dominant strategy. No matter what their opponents do, their best shot is to bid v_i. This also happens to maximize social welfare, as the max value player will receive the item. No other player will outbid their bid in order to avoid winning the item and receiving negative utility.

General Extension: Multiple Items

The natural extension of this problem is to generalize the second-price auction to multiple items. In such an auction, there are n players/bidders, and each has a value *function* $v_i(A)$ which represents the value of some set of items A to player i. We would like to see two things extended from the single-item case: the outcome should maximize social welfare, and bidding truthfully is the dominant strategy.

One issue is that in order for bidding truthfully to be the dominant strategy, it needs to be possible. So, we must adjust the bidding space so that players bid on packages of items rather than items individually. Two problems arise from this. Writing bids for every possible subset of the items on auction can be time consuming, since there an exponential number of subsets. Also, if items on auction are sold by multiple sellers (as with modern auctions like eBay), then this system can be logistically problematic or impossible. As a result, this interesting extension may not work so well in all applications.

In the meantime, we focus on the quality of maximizing social welfare, and hope that the players are bidding their true values. We would like to partition the set of the auctioned items to each player such that the social welfare is maximized:

$$\text{Outcome} = \max_{\text{partition } A_1 \ldots A_n} \left(\sum_i b_i(A_i) \right)$$

However, finding this partition can easily become NP-Hard. If $v_i(A) = 1$ when $A = A_i$ fixed and 0 otherwise, finding the partition becomes the NP-Complete set packing problem. However, this method should be done whenever possible.

VCG (Vickrey, Clarke, Grove) Mechanism

We introduce the VCG mechanism. This mechanism allows us to define a price that makes the bidding behavior truthful. We define the price for person i to not be related to what they want, but rather the

damage they cause to the rest of the bidders. In order to measure this price, let us assume an optimum partition exists:

$$A_1^*...A_n^* = \text{argmax}_i \left(\sum_i b_i(A_i) \right)$$

This damage that i causes can be expressed as the difference between the maximum welfare if i did not exist and the welfare they actually get when i is there. This gives us the following expression for price p_i:

$$p_i = \max_{A_1 \ldots A_n} \sum_{j \neq i} b_j \left(A_j \right) - \sum_{j \neq i} b_j \left(A_j^* \right) \tag{1}$$

Theorem 1. *VCG using the price formula in Eqn. 1 is truthful.*

Proof. The utility to player i is given by the difference between the value and the price:

$$\begin{aligned} u_i &= v_i(A_i^*) - \left[\max_{A_1 \ldots A_n} \sum_{j \neq i} b_j \left(A_j \right) - \sum_{j \neq i} b_j \left(A_j^* \right) \right] \\ &= \left(v_i(A_i^*) + \sum_{j \neq i} b_j \left(A_j^* \right) \right) - \max_{A_1 \ldots A_n} \sum_{j \neq i} b_j \left(A_j \right) \end{aligned}$$

Note that our final term in the expression above is independent of i. No matter what i bids, that term will stay the same. However, i can affect $A_1^*...A_j^*$, and therefore when the algorithm maximizes the utility of i, it will do so by maximizing the first two terms above. Therefore, when $v_i = b_i$, the algorithm and i's incentives are fully aligned. ■

Search Auction

We now consider a concrete auction in which multiple items are available: a *search auction*. The premise is that there are multiple slots numbered $1, \ldots, k$ available for advertisements at the top of the search results returned by a search engine, with slot i routing α_i web visitors to any advertisement placed there. We assume $\alpha_1 \geq \cdots \geq \alpha_k$. Players (or advertisers) $1, \ldots, n$ bid for the chance of advertising in one of the slots, with player i having value v_i for any web visitor directed from their ad.

There are multiple ways to allocate the slots to the players based on their bids; we have already seen the generalization of the first-price auction to multiple-item auctions. Here, we examine the so-called *generalized second-price auction* (GSP). In this auction format, each player submits a single bid, with the highest bidder obtaining the first slot and paying the price bid by the second-highest bidder, who obtains the second slot and pays the price bid by the third-highest bidder, and so on. The prices are per-click, so if the ith-highest bidder pays price p, their utility u_i is $\alpha_i(v_i - p)$. Note that for this to make sense we take $k < n$.

We explore the properties of GSP with an example:

$k = 2$: $\alpha_1 = 10$; $\alpha_2 = 7$
$n = 3$: $v_1 = 5$; $v_2 = 4$; $v_3 = 2$.

Let us first suppose that each player bids their true value. Then player 1 gets slot 1 for price 4 and player 2 gets slot 2 for price 2. Therefore,

$u_1 = 10(5 - 4) = 10$
$u_2 = 7(4 - 2) = 14$
$u_3 = 0$.

However, we readily see that player 1 has an incentive to deviate: given the other bids, if player 1 bids 3 instead of 5, they get slot 2 for price 2, in which case $u_1 = 7(5 - 2) = 21 > 10$.

Thus bidding one's true value is not a dominant strategy in general, so GSP is not a truthful mechanism. This is not the case in the second-price auction, rendering the term *generalized second-price auction* a misnomer. While both entail paying some next-highest bid, a key difference between the two is that in a second-price auction, if one decreases their bid below someone else's, they cannot obtain the sole item on auction, inducing a utility of 0; in a generalized second-price auction, the $k - 1$ highest bidders can decrease their bids below another person's bid yet still obtain a slot, just a lower one that brings less web visitors. However, their price decreases to the next lowest bid. This introduces a trade-off between gaining visitors and paying for the ad, so utility is not necessarily maximized by bidding as high as would be credible (i.e. their true value), which is the case in a second-price auction.

It may seem curious, then, that GSP would still be used instead of VCG for this auction type, which could also be implemented here. While the revenue of the auction with VCG depends on how advertisers value subsets of slots (likely the same as the highest slot in the subset), we at least know that VCG maximizes social welfare (i.e. of the advertisers). GSP is still often used in practical settings (e.g. by Google); fortunately Edelman, Ostrovsky, and Schwarz showed that GSP has a price of anarchy bound of 2. We may return to this fact in the last week of class.