# 4/6 Lecture Notes, CS 6840

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## Auctions

- Moving on from routing and location games, we consider auctions, where multiple people each compete with each other to buy a desired item or items. Today we consider the case with only one item, but we can extend this to multiple items, considered either simultaneously or sequentially.
- Setup: we have $n$ participants, each of which wants an item which has value $v_i$ to player $i$. Each player puts forward a bid $b_i$ for the item, which represents their "willingness to pay".
- We'll consider three models of payment: first price, second price, and all pay.
  - First price: the winner of the auction $i = \max_i b_i$ pays $b_i$ and takes the item, for a total utility of $v_i - b_i$; all other players pay nothing.
  - Second price: the winner $i^*$ pays the second-highest price $\max_{j \neq i} b_j$ and takes the item, for utility $v_i - b_j$; all other players pay nothing.
  - All pay: each player pays their bid, but only the highest bidding player takes the item: the winner's utility is $v_i - b_i$, and all other players have utility $-b_j$.
- Each payment model forms a different game and we consider the corresponding Nash equilibria.
  - For second price, a pure Nash equilibrium is achieved when each player sets $b_i = v_i$. To see this for an arbitrary player $i$, we only need to know the highest bid other than player $i$'s: if this bid $b_j$ is greater than $v_i$, then player $i$ won't want to win since she will be forced to pay $b_j$ which is greater than the value she places on the item, meaning her utility would be negative. If $b_j$ is less than or equal to $v_i$, then she will want to win the auction, so any bid above $b_j$ will be good enough to win her the item, and it doesn't matter how much greater than $b_j$ it is; since $b_j \leq v_i$, if she bids $b_i = v_i$ then she will lose when $b_j > v_i$ and win when $b_j \leq v_i$, as desired.
  - However, note that this is not a unique Nash equilibrium. Suppose the highest value among the bidders is $v_i = V$: then another Nash is a bid of $b_j = V + 1$ by any other player and 0 by all other players (including the highest value player). If any player other than $j$ wants to displace him, they will need to bid above $V + 1$, which means they will pay more than the item is worth; from their perspective there is no reason to bid anything above 0 when they won't win; so player $j$ wins the item and pays nothing.
  - All pay payment games have no pure Nash equilibria. This is because any player who isn't winning would prefer to lose nothing, so they reset their bid to 0; but once all players but one have bid zero, the remaining player can also set her bid to zero or some small amount and at this point, any other player can now increase their bids to win the item without exceeding their value cap. There's always at least one player who, given knowledge of all other players' bids, will want to change theirs no matter what the bids are set to be.
  - Finally, first price payment games do have pure Nash equilibria, but they are hard to find and calculate so they are generally analyzed as Bayesian games instead. In this case, each bidder know their own value and a probability distribution of all other players' values and attempts to maximize their expected gains with respect to the other players' random values and bids.