| CS 6840: Algorithmic Game Theory | Spring 2017 |
|---|---|
| Lecture 39: May 5 | |
| *Lecturer: Éva Tardos* | *Scribe: Dragos-Florian Ristache* |

## 39.1 Notes about Problem 3 from Problem Set 4

The question was about applying the VCG mechanism on the ad slot assignment problem. As a reminder, if the slots have click probabilities $\alpha_1 \geq \alpha_2 \ldots \geq \alpha_n$ and the bids per click are $b_1 \geq b_2 \ldots \geq b_n$, then the player who placed bid $b_i$ gets the slot $\alpha_i$.

A first observation is that while the optimization seems like a potentially complicated matching problem, where the cost of the edge from player i to slot j is $v_i \alpha_j$, the optimum welfare is actually obtained simply when player i is matched with slot i, where the players are sorted so that $v_1 \geq v_2 \ldots \geq v_n$.

The problem asked to analyze what happens when there are errors in the announced probabilities.

The first property was that even when the error ratio is bounded by $1 + \epsilon$, the bids can be unbounded. If there are two slots with equal advertised probabilities, $\alpha_1 = \alpha_2$, and the first is correct while the second's true probability is $\alpha_2 * (1 - \epsilon)$, the top slot is better and the price for it in VCG is 0, so none of the players has any incentive to stop bidding more than the other.

In order for the bids to be bounded, the condition $\alpha_i \geq \alpha_{i+1}\gamma$ with $\gamma > 1$ was added. In this case, if slot $\alpha_j$ was won with bid b, a player who bids above b needs to pay at least $b\alpha_j\gamma$, and repeating this process will eventually lead to a price larger than the player's value.

For the price of anarchy part, the trick was to use the classic deviation bid of $v_i/2$, then prove $u_i(v_i/2, b_{-1}) \geq \alpha_j v_i/2 - b_j(b) * \alpha_j$ where j is the slot i would have obtained in the optimal solution and $b_j(b)$ is the winning bid for j. Summing over i we get $SW \geq Opt/2 - \sum_j b_j(b)\alpha_j$.

In the first part, we use that $b_j \geq v_i$, so the sum is smaller than the social welfare, leading to a PoA of 4.

In the second part, we bound $b_j$ by whatever limit $x$ we found in c and get a PoA bound of $2(1 + x)$

## 39.2 Prizes and maximum welfare

### 39.2.1 Prophet's inequality

We first discuss about **the secretary problem**: given n numbers one by one, decide which one to stop at in order to maximize the probability of it being the largest. Any other outcome is considered a loss. This is usually approached by looking at the first few values (more precisely $N/e$), deciding a threshold based on them (in this case the maximum value), then stopping the first time a value is larger than the threshold. The risks are that a larger value might occur after we stop or that the threshold we set is too large.

The problem we are discussing has a less strict objective: knowing distributions $F_1, \ldots F_n$ so that the values $v_i \sim F_i$ we want to maximize $E[v_i]$ where i is the index we stop at. We will solve this with the same threshold approach described above.

The connection to pricing is the **single item buy now** auction, where the seller sets a reserve price and the first buyer who has a larger value purchases it. The goal is to maximize the expected value of that buyer, so the expected welfare.

We now do what we always do when we need to fix things in order to find lower bounds: we set the threshold to half the expectation of the maximum value, let's call it $v_0$.

The expected revenue of the auction will simply be $v_0 P(\text{someone buys the item}) = v_0 q$, where $q = P(\text{someone buys the item})$.

The expected utility of player $i$ if he is the buyer is $(v_i - v_0)^+ P(\text{no other player buys the item})$. We notice that this is actually larger than $(v_i - v_0)^+ P(\text{no one buys the item}) = (v_i - v_0)^+ (1 - q)$.

The expected sum of player utilities will be $(1-q)\sum_i (v_i - v_0)^+ \geq (1-q) max(v_i - v_0)^+ \geq (1-q)(max(v_i) - v_0)$. We take expectation again and use the fact that $v_0 = E[max(v_i)]/2$ to get the expected sum of utilities $= (1-q)(2v_0 - v_0) = (1-q)v_0$.

Then the expected welfare will be sum of revenue plus player utility, which is at least $v_0(q + 1 - q) = v_0 = E[max(v_i)]/2$.