CS 6840: Algorithmic Game Theory	**Spring 2017**

Lecture 35: April 26

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35.1 Estimating Value Distribution in Bayesian Nash

35.1.1 Auction Setting

Players:

- n players
- Symmetric value distribution $\mathcal{F}$
- Players draw their values from $\mathcal{F}$ independantly
- Players use a symmetric Bayes-Nash bidding rule: $b_i = b(v_i)$

You:

- Observe b_i for all players
- Know the auction system

Goal: Estimate $\mathcal{F}$

35.1.2 Motivation

Imagine that the players are experts at appraising the auction items, and you are not. You may want to learn a good estimate for the values of these types of items by observing the players.

Alternatively, imagine that you are an auctioneer considering switching to some new auction system. There is a significant time and opportunity cost for testing a new system and waiting for its behavior to reach an equilibrium. It would be better if you could make predictions about the behavior of the new system without testing it directly. In the previous part of the semester, we considered how to do this when the value distribution is already known. In real life, you first need to obtain that value distribution from the data.

Failing to take into account how underlying values affect bids can lead to bad results. Consider the following claim and purported proof, which are **FAKE NEWS**.

Claim 35.1 *GSP makes more money than VCG in ad-auctions.*

Proof: Say that $b_1 > b_2 > ...b_n$, where bidder i gets slot i with α_i click-rate.

Note that VCG and GSP have the same allocation rule, and differ only in payments. In GSP bidder i pays $\alpha_i b_{i+1}$. In VCG, it pays $\sum_{j>i} b_j(\alpha_{j-1} - \alpha_j) \leq b_{i+1}(\alpha_i - \alpha_n) \leq b_{i+1}\alpha_i$. ■

This is not a valid proof because while VCG is truthful, GSP is not, and so we should expect different (shaded) bids in GSP.

35.1.3 What auctions are good at inferring value distribution?

Definition 35.2 (Quantile Value) *$v(q)$ is the minimum value $\bar{v}$ such that $\Pr[v \leq \bar{v}] = q$, where v is drawn from $\mathcal{F}$.*

Assume v(q) is continuous. Additionally, define $b(q) = b(v(q))$, and let $x(q)$ be the expected allocation for players bidding at quantile q.

Today we will consider first price style formats. That is, assume all winners pay their bid. Then for a player whose value is $v(q)$, the utility of bidding as if their value was $v(s)$ is $(v(q) - b(s))x(s)$.

Assuming $b(q)$ is the Bayes-Nash equilibrium bid, then

$$q = \operatorname*{argmax}_s [(v(q) - b(s))x(s)]$$

To solve this, we set set the derivative to 0:

$$b'(s)x(s) = (v(q) - b(s))x'(s)$$

The bidding is at equilibrium if this maximum occurs at $s = q$, so if we have:

$$b'(q)x(q) = (v(q) - b(q))x'(q)$$

$$v(q) = b(q) + b'(q)\frac{x(q)}{x'(q)}$$

This formula allows us to infer the values from the bid distribution. Given the bid distribution, we can observe $b(q)$, which is the qth quantile of the bids, as well as $b'(q)$, the derivative of the bid as a function of the quantile. The rule for x depends on the auction format, so the value of $\frac{x(q)}{x'(q)}$ does too. This is known as the function of the auction format. Higher values of this ratio make the estimation harder, as the estimated value is more sensitive to the estimate of $b'(q)$.

Let's try out a few formats to see we get for this ratio.

As a special case, consider a first-price auction. By definition, a randomly drawn value is less than $v(q)$ with probability q. Since a bidder wins the item precisely when all other players submit lower bids (and thus drew lower values), this occurs with probability $x(q) = q^{n-1}$, and hence $x'(q) = (n-1)q^{n-2}$, and so the ratio is $\frac{q}{n-1}$.

As a second example, suppose we are running a first price version of the ad-action: we order bidders by their bid, and put bidder i in slot i. If we have slot i with $\alpha_i = (n-i)\epsilon$ probability of clicks, we get the following function for $x(q)$

$$x(q) = \sum_{i=0}^{n} i\epsilon \binom{n-1}{i} q^i (1-q)^{n-1-i}$$

as the bidder with value $v(q)$ gets the slot with $i\epsilon$ clicks (which is slot $n-i$), if there are $n-i-1$ bidders with higher value and i bidders of smaller value. This expression looks hard to evaluate, but it will be useful to look at the function

$$f(x) = \sum_{i=1}^{n} \binom{n-1}{i} i x^{i-1} q^i (1-q)^{n-1-i}$$

as the value we want is $x(q) = \epsilon f(1)$. Now observe that it is not hard to get the integral $F(x) = \int_0^x f(\xi) d\xi$, which is

$$F(x) = \sum_{i=1}^{n} \binom{n-1}{i} x^i q^i (1-q)^{n-1-i} = (xq + (1-q))^{n-1} - (1-q)^{n-1}$$

Using this fact, we get that $f(x) = (n-1)(xq + (1-q))^{n-2} q$, so we get $x(q) = \epsilon f(1) = \epsilon(n-1)q$, and so $x'(q) = \epsilon(n-1)$, and the ratio $\frac{x(q)}{x'(q)} = q$. This is a factor of $(n-1)$ bigger than what we had for the first price auction, so it makes the estimation easier.