CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 34: April 24

Lecturer: Éva Tardos *Scribe: Aaron Chen*

Administrative Stuff

Remaining coursework:

- PS4 (due Wed 4/26)
- Project (due 5/10)
- Take-Home Final (due by the end of Finals period, 72-hour sign out). Try to avoid the weekend since Éva and Theodoros will be out of town.

Office Hour Schedule for this week:

- Monday 2:30-3:30
- Tuesday (Theodoros) 5:30-6:30
- Wednesday 10:30-12

Finally, Bobby Kleinberg and Sid Banerjee will be guest lecturing this Friday and next Monday, respectively.

34.1 Effect of Budgets in Auctions

34.1.1 Introduction

We have not covered this thoroughly, but this appears as a question on PS4. When you have online auctions, they usually ask you for your budget, B_i (the total amount you're willing to pay per day) as well as a bid b_i (the amount you're willing to pay per click).

In an alternative (simplified) system, they only ask for your budget B_i. Available ads are shown proportionally to the budget.

34.1.2 A simple model

Let u be the number of items on sale (more generally a random variable). Each bidder is asked for their budget B_i. We get

$$x_i = \frac{B_i}{\sum_j B_j} u.$$

We must also include the utility of each player. In the simplest form, we can have linear utility functions $u_i(x) = a_i x$, $a_i \geq 0$. More generally, we can have $u_i(x) \geq 0$, continuous, and concave, with all the analysis preserved.

Now given B_j, $j \neq i$, player i wants to find the expression

$$\max_{B_i} u_i(x_i) - B_i, \quad x_i = \frac{B_i}{\sum_j B_j} u.$$

Appealing to single variable calculus,

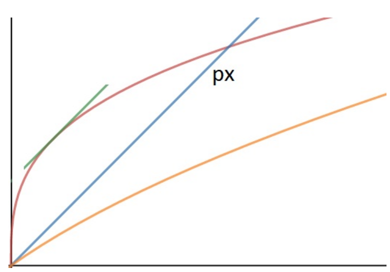
$$\begin{aligned} u_i'(x_i)x_i' - 1 &= 0 \\ u_i'(x_i)\left[\frac{1}{\sum_j B_j} - \frac{B_i}{(\sum_j B_j)^2}\right]u &= 1 \\ u_i'(x_i)\left[1 - \frac{B_i}{\sum_j B_j}\right] = \frac{\sum_j B_j}{u} &= p \text{ (price)}. \end{aligned}$$

where we call the last fraction price p, as that is the unit price the item is sold: a total of $\sum_j B_j$ paid for u items.

We see that $u_i'(x_i)(1 - \frac{x_i}{u}) = p$. In the special case that we assume that u is very large (relative to x_i), we see that approximately $u_i'(x_i) = p$.

Claim 34.1 *We claim that $u_i'(x_i) = p$ is the solution to the ith optimization problem for fixed price p, assuming that $x_i \geq 0$ for the solution).*

Proof: We want to maximize $u_i(x) - px$. This occurs when our utility is has slope equal to p.



■

We note that there can be cases where no solution exists, namely when our utility is bounded below by px for all $x > 0$ (alternatively when $u_1'(0) < p$), as seen in the yellow graph above.

Comparing the fixed price p optimization to the actual player optimization in Nash we also note that if x_i is big relative to u, then bidding like in the above analysis (till $u_i'(x_i) = p$) will drive up the price, so that should be compensated for if you are bidding for a large amount.

Thus, assuming that $B_i \geq 0$ and the solution exists, we bid according to the above analysis, and otherwise we bid 0.

34.2 Price of Anarchy for full info, pure strategy Nash

Recall the setup: we have bids $B_1, ..., B_n$ are Nash if

$$u_i'(x_i)\left(1 - \frac{x_i}{u}\right) = p, \quad x_i = \frac{B_i}{\sum_j B_j}u, \quad \text{and } p = \frac{\sum_j B_j}{u}.$$

Without loss of generality, let us make some simplifications:

1. Assume we have affine utility functions $u_i(x) = a_i(x) + b_i$.

   Proof: Consider $u_i(x)$ and the Nash amount x_i. Take the tangent line to u_i at Nash, and call it $\bar{u}_i(x)$. Observe that the Nash is unchanged by definition ($\bar{u}_i{}'(x_i) = u_i'(x_i)$), so some solution x_i satisfies the Nash condition with u_i if and only if it satisfies ith with $\bar{u}_i$.

   Finally, note that because $\bar{u_i'} \geq u_i$ for all x, Opt can only increase. ■

2. We can assume that $b_i = 0$, so we define $\tilde{u}(x) = a_i x$.

   Proof: Note that this leaves both the Nash and Opt unchanged, and we see that we have

   $$\frac{\text{Nash}}{\text{Opt}} \to \frac{\text{Nash} - b_i}{\text{Opt} - b_i}.$$

   The ratio only gets worse, so we can make this assumption. ■

Now, assuming that $u_i(x) = a_i x$ for all i, the price of anarchy bound we prove will clearly suffice for the general case. For convenience, assume that $a_1 \geq a_2 \geq ... \geq a_n$.

Note that the optimal solution would be for player 1 to buy everything ($x_1 = u$), leading to total utility $a_1 u$. Now suppose $x_1 < u$. Looking at the Nash for player 1, we have $a_1(1 - \frac{x_1}{u}) = p$.

Now consider other users. A user with $x_i > 0$ most have $a_i(1 - \frac{x_i}{u}) = p$, which implies that $a_i > p$. Now with many other users who all have $a_i \approx p$ (just above p), we see that the total utility in this case is $x_1 a_1 + (u - x_1)p$. Then the total utility divided by the Opt is

$$\frac{x_1 a_1 + (u - x_1)p}{a_1 u} = \frac{x_1 a_1 + (u - x_1)a_1(1 - \frac{x_1}{u})}{a_1 u}.$$

substituting the expression for price p. Now canceling the a_1 and substituting $y = \frac{x_i}{u}$, the expression simplifies to $y + (1 - y)(1 - y) = y^2 - y + 1$. This is minimized when $2y = 1$, or $y = 1/2$. Back substituting, we see that the expression above greater than $3/4$. So we have proved the following:

Theorem 34.2 *For utilities u_i concave, continuous, and non-negative, the Price of Anarchy of fair sharing at pure Nash is bounded above by 4/3.*

Our analysis also gives us the worst case scenario where one person bids about half of the market share which ruins the price for everyone else.