28.1 Learning in Multi-Item Simultaneous Auctions

Suppose that there are \( n \) bidders and \( m \) items. Denote the set of all items by \( S \). A bid \( b_{ij} \) for player \( i \) specifies an item and a bid for that item. In this setup there are uncountably many strategies available to a bidder. In order to apply the multiplicative weights algorithm, we need to discretize the possible bids that agents can report. For simplicity, suppose that \( v_{ij} \in [0, 1] \) for all \( i \) and \( j \). Bids are then \( b_{ij} \in \delta \mathbb{Z} \); i.e. agents may bid in increments of \( \delta \). How does discretizing bids affect regret? Should bids be rounded up or down to fit the discretization? The latter question has no clear-cut answer. Agents could round bids down and risk not receiving goods while paying a lower price, or agents could round up and pay a higher price for a better chance at a bundle. In particular, agents could receive negative payoffs by rounding their bids up (if \( \delta > v_{ij} \)). This is really an issue for coarse discretizations, so making \( \delta \) sufficiently small should minimize this issue. In this lecture, we’ll assume that agents round their bids up so that a reported bid is converted as: \( b_{ij} \to \delta \left\lceil \frac{b_{ij}}{\delta} \right\rceil \)

Rounding bids in this way can yield an additional regret of at most \( m\delta \) for any given bidder (an extra regret of \( \delta \) is possible for every item \( j \)). Despite solving the problem of uncountably many alternatives to bidders, this specification still yields two undesirable features.

- Running time per step: \( \left( \frac{1}{\delta} \right)^m \); e.g. \( \delta \sim \frac{1}{m} \Rightarrow \) running time \( m^m \) per step
- Full information regret: \( \frac{m \log m}{\epsilon} \)

Notice that the running time per step can be quite large if there are many items and a fine discretization. In this lecture, we’ll consider a less demanding bid structure.

Suppose that agents bid a single number \( b_i \) for all items. Players are ordered according to decreasing bids and sequentially choose their favorite items from what remain, paying price \( b_i \) for each item that they claim.

Under these conditions, the multiplicative weights algorithm has the following features:

- Running time per step: \( \frac{1}{\delta} \)
- Full information regret: \( \frac{\log \frac{1}{\epsilon}}{\epsilon} \)
- Partial information regret: \( \frac{1}{\epsilon \delta} \)

These results will be useful in the next lecture.
28.2 Price of Anarchy Bounds

Consider the following simple case: bidder \( i \) has a valuation \( v_i \) for all items \( j \in S_i \subseteq S \). This implies \( v_i(S) = v_i|S \cap S_i| \).

**Theorem 28.1** Suppose agents submit a single bid for all items without rounding. Then the auction has a price of anarchy bound of 2 in the simple case above.

**Proof:** The result follows from smoothness. In particular, we seek to show that the game is \((\frac{1}{2}, 1)\) smooth. Let \( b_i \) be a vector of bids for each player. Since this is a first-price setting, the optimal bid for agents is \( b_i^* = \frac{v_i}{2} \). Let \( S_i^* \subseteq S_i \) be the items that \( i \) receives in the optimum. We seek to show that

\[
\frac{1}{2} v_i - \sum_{j \in S_i^*} p_j(b) \geq 0.
\]

When bidder \( i \) bids \( b_i^* \) on some item \( j \in S_i^* \), there are two possible outcomes:

- \( i \) wins \( j \) with \( b_i^* \) and receives utility \( \frac{v_i}{2} \)
- \( i \) does not win item \( j \), which implies that \( i \) gets utility 0 from item \( j \) and some other agent paid a higher price so that \( p_j(b) \geq b_i^* = \frac{1}{2} v_i \)

In the first case, the result holds with equality for good \( j \) before subtracting the price \( p_j(b) \). In the second case, we have that the agent receives no utility from good \( j \) and \( \frac{1}{2} v_i - p_j(b) \leq 0 \). Summing over all goods, we have that

\[
\frac{1}{2} v_i - \sum_{j \in S_i^*} p_j(b) \geq 0.
\]

Thus the game is \((\frac{1}{2}, 1)\) smooth, and so the price of anarchy bound is 2.

The above theorem showed a price of anarchy bound for a very simple specification of agent valuations. Now, suppose that set valuations are fractionally subadditive so that \( v_i(S) = \max_k \sum_{j \in S} v_{ij}^k \). Recall that \( k \) in this setting denotes agent \( i \)'s usage of good \( j \). Allowing valuations to be fractionally subadditive raises the price of anarchy for this auction.

**Theorem 28.2** Suppose valuations are fractionally subadditive and that agents still bid a single value for all items they desire. Then the price of anarchy for this auction is bounded by \( 2H_m \), where \( H_m \) is the \( m \)th harmonic number.

**Proof:** Again, letting \( S_i^* \) be the set of goods that bidder \( i \) receives at the optimum, we have that

\[
v_i(S_i^*) = \max_k \sum_{j \in S_i^*} v_{ij}^k = \sum_{j \in S_i^*} v_{ij}^{k_i}
\]

so \( k_i \) is the additive valuation that determines the bidders value for her optimum bundle.

We want to find a possible bidding strategy \( b_i^* \) for \( i \), so that not-regretting \( b_i^* \) gives us the price of anarchy bound. Define \( v_{i\mu}^{k_i} = \arg\max v_{ij}^k \{ l \in S_i^* : v_{ij}^k \geq v_{ij}^{k_i} \} \). Let \( b_i^* = \frac{v_{ij}^{k_i}}{2} \). When bidder \( i \) bids \( b_i^* \) in this auction, then for each item \( l \) such that \( v_{ij}^k \geq v_{ij}^{k_i} \), there are two possible cases:
• Bidder $i$ has the option of taking item $l$ with bid $b_i^*$ and hence $v_{il}^{k_i} - b_i^* \geq b_i^*$.

• Bidder $i$ does not have item $l$ available, which implies a utility of 0 from good $l$ and price $p_l(b) \geq b_i^*$.

Now consider the utility bid $b_i^*$ gets against the bid vector $b_{-i}$ of other bidders. Let $\hat{S}_i = \{ j : v_{ij}^{k_i} \geq v_{ij}^{k_i} \}$ and let $\hat{S}_i \subset \hat{S}_i^*$ denote the set of items still available in the auction when $i$ with $b_i^*$ gets to select items. Clearly, $i$ can select this set, which would give him utility

$$v_i(\hat{S}_i) - |\hat{S}_i| b_i^* \geq \sum_{j \in \hat{S}_i} (v_{ij}^{k_i} - b_i^*) \geq \sum_{j \in \hat{S}_i^*} (b_i^* - p_j(b))$$

where the last inequality follows as for items in $j \in \hat{S}_i$ we have $v_{ij}^{k_i} \geq 2b_i^*$, and for the remaining items in $\hat{S}_i$ are no longer available, so the price is above $b_i^*$.

Now in the auction, $i$ gets to select the subset of remaining items giving her maximum utility, so her utility is at least this big, so we get the bound

$$u_i(b_i^*, b_{-i}) \geq \sum_{j \in \hat{S}_i^*} (b_i^* - p_j(b))$$

If we could show that $2b_i^* |\hat{S}_i| = v_{ij}^{k_i} |\hat{S}_i| \geq \frac{v(S_i^*)}{H_m}$ then we could continue this as

$$u_i(b_i^*, b_{-i}) \geq \frac{1}{2H_m} v_i(S_i^*) - \sum_{j \in \hat{S}_i^*} p_j(b) \geq \frac{1}{2H_m} v_i(S_i^*) - \sum_{j \in \hat{S}_i} p_j(b)$$

showing that the auction is $(\frac{1}{2H_m}, 1)$-smooth, and hence has a price of anarchy of at most $2H_m$ as claimed. We'll prove this claim as a separate lemma.

To simplify the notation, we can drop $i$, and $k_i$ from the notation. What we need to prove is the following.

**Lemma 28.3** Given a vector of values $v_j$, and a set $S$ with $|S| \leq m$, define $\mu = \argmax_j v_j |l \in S, v_l \geq v_j|$, then

$$v_\mu(|j \in S : v_j \geq v_\mu|) \leq v(S) \leq H_m v_\mu(|j \in S : v_j \geq v_\mu|)$$

**Proof:** The first inequality true for any $\mu$ by definition. To see the second one, let $v_\mu(|j \in S : v_j \geq v_\mu|) = W$. Since $W$ is the maximum $W = \max_j v_j |l \in S, v_l \geq v_j|$, the maximum value of $v_j$ in the set $S$ is at most $W$, the second highest can be at most $W/2$, the third highest at most $W/3$, and the $t$ highest at most $W/t$, as there $S$ can have at most $t - 1$ elements with $v_j > W/t$ (or else the maximum would be higher). Using these bounds we get that

$$\sum_{j \in S} v_j \leq W + W/2 + W/3 + \ldots + W/m = H_m W$$

as claimed.