CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 26: March 29

Lecturer: Éva Tardos *Scribe: Horace He*

Today's lecture primarily covered proving the price of anarchy for multi-item, multi-player auctions with XOS (fractionally sub additive) valuations. We can also notice that this is a smoothness based proof, and so this implies a bound for full information Nash, Bayes Nash (assuming players are independent), and coarse correlated equilibria learning outcomes.

We also note that our learning algorithm for coarse correlated equilibria runs into some issues. We must maintain a weight for every decision (uncountably infinite as there is no limit for what we can bid). Even if we limit ourselves to say, integers, we still have an exponential amount of values we must keep track of (The Monday seminar talked about a method for which we can do it polynomially[1].)
We will cover simpler auctions where the bidding is learnable in future lectures.

26.1 Price of Anarchy for 1st Price Auctions

We currently have 2 frameworks for auctions: first price and second price. We will prove the price of anarchy for the first auction and then show the modifications needed for second price.

Proof: Remember that b is our bid vector, with b_{-i} the bids of all players other than i. We want to defined a special bid $b_i^*(v)$ for all players i (depending on the value vector) and want to prove that the following holds for all bid vectors b.

$$\sum_i u_i(b_i^*(v), b_{-i}) \geq \frac{1}{2}\text{Opt}(v) - \text{rev}(b)$$

As this is our smoothness equality for auctions, it would also imply that PoA ≤ 2.
Now suppose that player i wins S_i^* in the optimal solution with values v. Then,

$$\text{Opt}(v) = \sum_i v_i(S_i^*)$$

Then, we replace the revenue with $\sum_{j \in S_i^*} p_j(b)$, where $p_j(b)$ = price of item with bids b. This gives us the new equality for us to prove:

$$u_i(b_i^*(v), b_{-i}) \geq \frac{1}{2} v_i(S_i^*) - \sum_{j \in S_i^*} p_j(b)$$

Now, we know through XOS valuations that

$$v_i(S_i^*) = \sum_{j \in S_i^*} v_{ij}^{k_i}$$

where k_i is arg max of the definition of $v_i(S_i^*) = \max_k \sum_{j \in S_i^*} v_{ij}^k$.
We now define $b_{ij}^*(v) = \frac{1}{2} v_{ij}^{k_i}$ for all j, and let A be the set of items i with with bid b_i^* against the bids b_{-i}.

[1] http://www.cs.cornell.edu/courses/cs7890/2017sp/Haghtalab%203-27.html

Now we have the following

$$u_i(b_i^*(v), b_{-i}) = v_i(A) - \frac{1}{2}\sum_{j\in A} v_{ij}^{k_i}$$

This is his value on the items A (ie: what he wins and pays for). Now, we replace the first quantity on the right side with his value through XOS

$$\geq \sum_{j\in A} v_{ij}^{k_i} - \frac{1}{2}\sum_{j\in A} v_{ij}^{k_i}$$

Now, as all differences are non-negative in A, so we can drop the terms not in A, and for for $j \notin A$, $p_{ij} > \frac{1}{2}(v_{ij}^{k})$, so we can add those items

$$= \sum_{j\in A}\frac{1}{2}v_{ij}^{k_i} \geq \sum_{j\in A\cap S_i^*}(\frac{1}{2}v_{ij}^{k_i} - p_j(b)) \geq \sum_{j\in S_i^*}(\frac{1}{2}v_{ij}^{k_i} - p_j(b)) = v_i(S_i^*) - \sum_{j\in S_i^*} p_j(b)$$

as claimed. Summing over all players we get the required bound of

$$\sum_i u_i(b_i^*(v), b_{-i}) \geq \frac{1}{2}\mathrm{Opt}(v) - \mathrm{rev}(b)$$

This smoothness inequality implies the bound of 2 on the price of anarchy. ■

26.2 2nd price Auctions

One assumption we need to make is that there is no overbidding (that is, for any bid b_i and any subset A, $\sum_{j\in A} b_{ij} \leq v_i(A)$). However, this is not necessarily a rational behavior: it is true, if bidders are extremely afraid of getting negative utility.

Given the above assumption, what do we need to change for second price auctions?

- Replace revenue (p_j) with $\max_i b_{ij}$.
- Use $b_{ij}^* = v_{ij}^{k_i}$ with the same k_i instead of the definition above the 1/2 factor (no shading necessary as it's a 2nd price auction.)

Now, for 2nd price auctions, most of the framework remains the same.
Proof: Let A be the set of items i wins against the bid vector b_{-i} with our new bid b_i^*. We have that:

$$u_i(b_i^*(v), b_{-i}) = v_i(A) - \sum_{j\in A}\max_{\ell\neq i} b_{\ell j}$$

We impose the condition $\ell \neq i$ as it is a second price auction, so this is the second highest value.
Through XOS valuations, we substitute the bid value in to get

$$\geq \sum_{j\in A} v_{ij}^{k_i} - \sum_{j\in A}\max_{\ell\neq i} b_{\ell j}$$

Now, notice that for all $j \in A$ $v_{ij} \geq \max_{\ell\neq i} b_{\ell_j}$ as i wins item j with bid v_{ij} so we can drop all terms in $A \setminus S_i^*$. Similarly, for all $j \notin A$ $v_{ij} \leq \max_{\ell\neq i} b_{\ell j}$ as i losses this item with bid v_{ij} so we can add all terms in $S_i^* \setminus A$.

$$\geq \sum_{j\in A\cap S_i^*}(v_{ij}^{k_i} - \max_{\ell\neq i} b_{\ell j}) \geq \sum_{j\in S_i^*}(v_{ij}^{k_i} - \max_{\ell\neq i} b_{\ell j})$$

$$\geq v_i(S_i^*) - \sum_{j \in S_i^*} \max_{\ell} b_{\ell j}$$

Summing over all players, we can substitute $Opt(v) = \sum_i v_i(S_i^*)$ in.

$$\sum_i u_i(b_i^*(v), b_{-i}) \geq \mathrm{OPT}(v) - \sum_j \max_{\ell} b_{\ell j}$$

As there is no overbidding, we can replace the 2nd term on the right side with social welfare: let $S_i(b)$ the set of items player i wins with bids b, then the second term is

$$\sum_j \max_{\ell} b_{\ell j} = \sum_i \sum_{j \in S_i(b)} b_{ij} \leq \sum_i v_i(S_i(b))$$

giving us

$$\geq \mathrm{Opt}(v) - \mathrm{SW}(b)$$

Thus, we get the smoothness inequality of

$$\sum_i u_i(b_i^*, b_{-i}) \geq Opt(v) - SW(b)$$

implying a bound of 2 on the PoA. (Using that at an equilibrium no player regrets bidding b_i^*, so have utility at least $u_i(b_i^*, b_{-i})$, and now we can move social welfare to the left side, we see that our PoA is 2.) ■