Valuation Classes

Up to now we have had single item or unit demand where the value of a set \( v(S) = \max_{j \in S} v_j \). This value (as all the ones below) is what we normally call \( v_i \) (the value to one person).

24.1 Subadditive

If \( A, B \) are sets and \( v(A) \) and \( v(B) \) are the values of these sets then
\[
v(A) + v(B) \geq v(A \cup B)
\]
We assume this inequality always holds for this class, since without it it is difficult to do anything meaningful.

We will also assume that the value functions are normalized. So \( v(\emptyset) = 0 \) and \( v(S) \leq v(S') \) if \( S \subseteq S' \), i.e., there is free disposal. These two together also imply that \( v(S) \geq 0 \) for all \( S \).

24.2 Decreasing Marginal Utility

If \( S \subseteq S' \) and \( j \) is an item then
\[
v(S + j) - v(S) \geq v(S' + j) - v(S')
\]
Where \( v(S + j) - v(S) \) is the marginal utility of item \( j \) when added to set \( S \).

With the assumption that \( v(\emptyset) = 0 \), the subadditive inequality can be re-written in a form closer to this one:
\[
v(A) - v(\emptyset) \geq v(A \cup B) - v(B)
\]

**Theorem 24.1**  
**Decreasing Marginal Utility \( \implies \) Subadditive**

We propose that Decreasing Marginal Utility \( \implies \forall S \subseteq S' \text{ and } A, v(S \cup A) - v(S) \geq v(S' \cup A) - v(S') \). We prove this claim by induction on \( |A| \):

**Proof:** Say \( j \in A, A' = A \setminus \{j\} \) then by induction \( v(S \cup A') - v(S) \geq v(S' \cup A') - v(S') \). By definition \( S \cup A' \leq S' \cup A' \implies v(S \cup A' + j) - v(S' \cup A') \geq v(S' \cup A' + j) - v(S' \cup A') \). Now since \( A' + j = A \) we can see that this is the sum we wanted.

**Corollary 24.2**  
\( S = \emptyset \implies \text{Subadditive inequality where } B = S' \).
An alternative way of writing the Decreasing Marginal Utility inequality is

\[ v(A) + v(B) \geq v(A \cap B) + v(A \cup B) \]

for all sets \( A \) and \( B \).

In this form Decreasing Marginal Utility is called Submodular (means the same thing but used in different fields). **Proof:** Rearranging this we get

\[ v(A) - v(A \cap B) \geq v(A \cup B) - v(B) \]

which is the same as the equation in the proof above if \( S = A \cap B \) and \( S' = B \).

### 24.3 Fractionally Subadditive

Fractionally Subadditive is a version of Subadditive where you can take sets fractionally. So we now have a multiplier \( x_A \), sets \( A \), for all sets.

Set \( S \) is covered if \( \sum_{A : i \in A} x_A \geq 1 \forall i \in S \).

If \( S \) is covered by \( x \) then \( \sum_A x_A v(A) \geq v(S) \).

**Theorem 24.3** Fractionally Subadditive \( \implies \) Subadditive since \( x_A = x_B = 1 \) makes the set \( S = A \cup B \) covered.

### 24.4 XOS

This valuation class is algorithmically nice to use but looks very different than the others.

An additive valuation is defined by having value \( v_j \forall \) items \( j \). The total value of a set \( S \) before was \( v(S) = \sum_{j \in S} v_j \). Instead we now have multiple possible values for each item \( b^k_j \), and use

\[ v(S) = \max_k \sum_{j \in S} v^k_j \]

Given \( v^k_j \) for \( k = 1, \ldots, n \) on items, where the \( k \) values represent that the item may have different values depending on its different uses.

**Claim 24.4** unit demand is a special case of XOS

We have from earlier that unit demand uses \( v(S) = \max_{j \in S} v_j \). This function has no \( k \) so we must make a \( k \) to fit the function. We use

\[ v^k_j = \begin{cases} v_j & j = k \\ 0 & \text{otherwise} \end{cases} \]

So that we have the vector \( v_j^* = [0, \ldots, v_j, 0, \ldots, 0] \).

**Claim 24.5** XOS is Subadditive
Proof:
We define
\[ v(A \cup B) = \max_k \sum_{j \in A \cup B} x_j^k = \sum_{j \in A \cup B} v_j^{k^*} \]
that is, let \( k^* \) be the value where the maximum occurs for the set \( A \cup B \). Now we have
\[
v(A \cup B) = \sum_{j \in A \cup B} v_j^{k^*} \leq \sum_{j \in A} v_j^{k^*} + \sum_{j \in AB} v_j^{k^*} \leq \max_k \sum_{j \in A} x_j^k + \max_k \sum_{j \in B} x_j^k
\]
where the first inequality is true as the items in \( A \cap B \) are now included twice, and the second inequality is true as \( k^* \) is one possible value for the \( k \) in the max.

\[ \square \]

Claim 24.6 \( XOS \) is Fractionally Subadditive.

Proof:
Same as Subadditive proof above but now we have \( x_A \)
We have \( x_A \) sets and \( S \) is covered. So \( v(S) = \sum_{j \in S} v_j^{k^*} \), as before \( k^* \) is where the max occurs for set \( S \). Using this and other equations from above we get that
\[
\sum_A x_A v(A) = \sum_A x_A [\max_k \sum_{j \in A} v_j^k] \geq \sum_A x_A \sum_{j \in A} v_j^{k^*} = \sum_j v_j^{k^*} (\sum_{A,j \in A} x_A) \geq \sum_{j \in S} v_j^{k^*} = v(S)
\]
where the last inequality is true because \( S \) is covered so for \( h_j \in S \) we have \( \sum_{A,j \in A} x_A \geq 1 \).

\[ \square \]

Facts:
Fractionally Subadditive=\( XOS \)
Submodular \( \implies XOS \)
(Proofs may or may not be covered in a different lecture).