CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 19: March 10

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19.1 Auctions and Smoothness

19.1.1 Overview

So far, we've looked at single item auctions, with first price and all pay. Today we'll look at multiple items with *unit demand*, meaning every player wants exactly one item. Let v_{ij} be the value of item j to buyer i. For any set of items S, let $v_i(S) = \max_{j \in S} v_{ij}$. This is known as *free disposal*: if a buyer gets more items than he wanted, he can "throw away" the bad ones (although this still isn't great—he did have to pay for them!).

Now second price isn't as helpful because now the buyers don't know who's bidding on what item. Suppose the items are sold by different people (think eBay). In this case maybe buyers want to bid on more than one. But today, we're going to focus on the case of first price auctions where buyers are only allowed to bid on one item.

19.2 Classical Results

19.2.1 Symmetric Case

First we consider the symmetric special case where $v_{ij} = 1$ for all i and j, and the number of items is equal to the number of buyers. The first possible equilibrium is when everyone coordinates. Everyone is assigned a distinct item to bid on, and they all bid ε. The social welfare in this case is n. The other equilibrium is when each player chooses their item uniformly at random. Exactly how much to bid should be random as well, but requires some calculus which we won't get into. This equilibrium is definitely worse since now some items could be totally unused. Since social welfare includes the revenue, any item sold contributes 1 to the welfare. Thus

$$\mathbb{E}[\text{welfare}] = n - \mathbb{E}[\#\text{ items not bid on}] \approx n - \frac{n}{e} = (1 - \frac{1}{e})OPT$$

since

$$\Pr[\text{no one bids on item } j] = \left(\frac{n-1}{n}\right)^n \approx \frac{1}{e}.$$

19.2.2 General Case

Now we go back to the general case, where $v_{ij} \geq 0$. Suppose we have a distribution σ which is a full-information mixed Nash (meaning all v_{ij} are fixed). Then

$$\mathbb{E}_{b\sim\sigma}[u_i(b)] \geq \mathbb{E}_{b\sim\sigma}[u_i(b_i^*, b_{-i})]$$

for all bids b_i^*. Find the optimum solution, and define

$$X_{ij}^* = \begin{cases} 1 & \text{if buyer } i \text{ gets item } j \text{ in OPT} \\ 0 & \text{else.} \end{cases}$$

In this case, the social welfare of the opt is $\sum_{ij} v_{ij} X_{ij}^*$.

Our above inequality held for all b_i^*, so in particular let's define b_i^* to be the bid where i bids 0 on all items expect the item j where $X_{ij}^* = 1$ and bids $v_{ij}/2$ for item j. We can now get two helpful inequalities. For all j,

$$u_i(b_i^*, b_{-i}) \geq \frac{1}{2} v_{ij} - p_j(b)$$

where $p_j(b)$ is the price paid for j in bids b. This is because if i wins with b_i^*, then $u_i(b_i^*, b_{-i}) = v_{ij} - b_i^* = \frac{1}{2} v_{ij}$, and if i loses with b_i^*, then $p_j(b) \geq b_i^*$ and $u_i(b_i^*, b_{-i}) = 0$. We can also sum to get

$$u_i(b_i^*, b_{-i}) \geq \sum_j \left(\frac{1}{2} v_{ij} - p_j(b) \right) X_{ij}^*$$

since at most one of these summands will be non-zero. This is because buyer i is only bids on one item in b_i^*, the one of the $X_{ij}^* \neq 0$. Now we can compute to see

$$\begin{aligned} \mathbb{E}[\sum_i u_i(b)] &\geq \mathbb{E}[\sum_i u_i(b_i^*, b_{-i})] \\ &\geq \mathbb{E}[\sum_i \sum_j \left(\frac{1}{2} v_{ij} - p_j(b) \right) X_{ij}^*] \\ &= \frac{1}{2} \sum_{ij} v_{ij} X_{ij}^* - \mathbb{E}[\sum_j p_j(b) \sum_i X_{ij}^*] \\ &\geq \frac{1}{2} OPT - \mathbb{E}[\sum_j p_j(b)] \\ &= \frac{1}{2} OPT - \mathbb{E}[rev(b)]. \end{aligned}$$

Rearranging, we get

$$\mathbb{E}[\sum_i u_i(b) + rev] \geq \frac{1}{2} OPT$$

and so we have shown that the price of anarchy is $\geq 1/2$.

We note that if we use a randomized bid as b_i^* for the item j with $X_{ij} = 1$ with density function $f(x) = \frac{1}{v_{ij} - x}$ in $[0, (1 - \frac{1}{e} v)]$ then one can prove that the resulting value in expectation is at least a $(1 - \frac{1}{e})$ fraction of the optimum. It is interesting to note that this is the best possible bound, as we have seen by the simple example above.

We now have two issues to consider: why require everyone to bid on just one item? And what kind of info do the players really need?

19.3 Recap

In the single item first price case, we had $b_i^* = v_i/2$, and price of anarchy $1 - 1/e$ (we saw $1/2$). This is true even if $(v_1, \ldots, v_n)$ is random, coming from a correlated distribution.

In single item all pay, and in the unit demand actions, we used a b_i^*, that depends on $v_1, \ldots, v_n$ (since not max means you need to not regret bidding 0 in all pay, and being max value means you need to not regret a randomized bid). The resulting price of anarchy bounds of 1/2, works for Nash but NOT for the correlated Bayesian version—if the other buyer's values are random, now you don't know how to bid, or more precisely the b_i^* we used is not well defined.

Soon we'll talk about learning outcomes and coarse correlated equilibriums, and we'll prove that the price of anarchy bound is also true when the values come from independent distributions.