Lecture 17: March 6

Lecturer: Éva Tardos — *Scribe: Derek Lougee*

All-pay Auctions

There are n players with valuations $v_1, \ldots, v_n$ drawn from a distribution F. Each player places a bid, $b_1, \ldots, b_n$ and pays the value of their bid whether they win or lose. The player with the maximum bid wins and receives utility equal to their valuation. That is, if player i wins, their utility is $u_i = v_i - b_i$, and if they lose it is $u_i = -b_i$.

17.1 Incomplete Information

Consider a simple example with $n = 2$ and v_i independent and identically distributed according to a uniform distribution on $[0, 1]$. We are interested in finding a monotone increasing bidding function, $b(v)$, that constitutes a symmetric Nash equilibrium. Note that if a player i bids $b(x)$, their probability of winning is $P(b(x) > b(v_j)) = P(v_j < x) = x$ since b is assumed to be monotone increasing. Thus if player i bids as if their valuation were x, their utility is

$$u_i(x|v_i) = v_i x - b(x).$$

In order for b to be a Nash equilibrium, it must be that player i's utility is maximized by choosing $x = v_i$. That is, from the first-order condition at $x = b_i$, $v_i = b'(v_i)$. Integrating,

$$b(v_i) = \frac{v_i^2}{2}.$$

Since this function is indeed monotone increasing, it constitutes a Nash equilibrium.

The revenue from the all-pay auction in this example is $R = b(v_1) + b(v_2) = \frac{v_1^2}{2} + \frac{v_2^2}{2}$. Since v_1, v_2 are i.i.d. $U[0, 1]$, expected revenue is

$$E(R) = E\left(\frac{v_1^2}{2} + \frac{v_2^2}{2}\right) = E(v^2) = \int_0^1 v^2 \, dv = \frac{1}{3}.$$

Recall that for this example, the symmetric Nash bidding function for a first-price auction is $b(v) = \frac{v}{2}$, so the revenue in a first-price auction is $R_1 = \max\left\{\frac{v_1}{2}, \frac{v_2}{2}\right\}$. Since v_1, v_2 are i.i.d. $U[0, 1]$,

$$P(\max\{v_1, v_2\} \le x) = P(v_1 \le x)P(v_2 \le x) = x^2.$$

Thus,

$$E(R_1) = \frac{1}{2}E(\max\{v_1, v_2\}) = \frac{1}{2}\int_0^1 x \cdot (2x) \, dx = \frac{1}{2}\frac{2}{3} = \frac{1}{3}.$$

The symmetric Nash bidding function for a second-price auction is $b(v) = v$, so the revenue in a second-price auction is $R_2 = \min\{v_1, v_2\}$ (since the winner pays the second highest bid). Again since v_1, v_2 are i.i.d. $U[0, 1]$,

$$P(\min\{v_1, v_2\} \le x) = 1 - P(\min\{v_1, v_2\} \ge x) = 1 - (1 - x)(1 - x) = 2x - x^2,$$

and so

$$E(R_2) = E(\min\{v_1, v_2\}) = \int_0^1 x \cdot (2 - 2x)dx = \frac{1}{3}.$$

That is, all-pay, first-price, and second-price auctions all have the same revenue in expectation, a fact that is referred to as *revenue equivalence*.

17.2 Full Information

Recall that there is no Nash equilibrium in pure strategies in an all-pay auction with full information. Note that any player that loses the auction must bid 0, and so the winner will also bid 0. However, if the winner is bidding 0, any loser with a positive valuation would deviate to a higher bid and win the auction. Thus we will consider a Nash equilibrium in mixed strategies.

Claim. In the case of two players with valuations $v_1 = v_2 = 1$, bidding b_i according to a uniform $[0, 1]$ distribution constitutes a Nash equilibrium.

Proof: Suppose player i deviates and bids $x \in [0, 1]$ and player j follows the randomized bidding strategy. Then i's probability of winning is $P(b_j < x|x) = x$ since $b_j \sim U[0, 1]$. Thus player i's pay-off from bidding x is

$$u_i(x|v_i) = v_i x - x = 0$$

since $v_i = v_j = 1$. That is, player i's utility is constant in x so randomizing over $[0, 1]$ is a best response. ■

The revenue of the all-pay auction in this setting is $R = b_1 + b_2$ so

$$E(R) = E(b_1) + E(b_2) = \frac{1}{2} + \frac{1}{2} = 1.$$

In a first- or second-price auction, both players bid $b_1 = b_2 = 1$ so $R_1 = R_2 = 1$. That is, revenue equivalence still holds.

Fact. Revenue equivalence does not always hold without the symmetry of the bidders.

Consider the case of two players with $v_1 = 2$ and $v_2 = 1$. In a first-price auction, the Nash equilibrium in pure strategies is to bid $b_1 = b_2 = 1$; in a second-price auction, the Nash equilibrium in pure strategies is to bid $b_1 = 2$, $b_2 = 1$. In both cases, the revenue to the seller is 1.

In the case of an all-pay auction, there is no Nash equilibrium in pure-strategies. Consider the following strategies: player 1 bids $b_1 \sim U[0, 1]$ and player 2 bids $b_2 = 0$ with probability $\frac{1}{2}$, and with probability $\frac{1}{2}$ bids $b_2 \sim U[0, 1]$. This is a Nash equilibrium. If player 1 bids $x \in [0, 1]$ (clearly player 1 would never bid $x > 1$ since $v_2 = 1$ implies $b_2 \leq 1$), their probability of winning is $\frac{1}{2}$ if $b_2 = 0$ plus $\frac{1}{2}x$ when b_2 is drawn from $U[0, 1]$. Therefore player 1's utility of bidding x is

$$u_1 = 2\left(\frac{1}{2} + \frac{1}{2}x\right) - x = 1.$$

Since player 1's utility is constant in their own bid, randomizing is a best response. If player 2 bids $x \in [0, 1]$, their probability of winning is simply x since $b_1 \sim U[0, 1]$ and so

$$u_2 = 1 \cdot x - x = 0.$$

The expected revenue of the all-pay auction is then

$$E(b_1 + b_2) = \frac{1}{2} + \left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{3}{4}.$$

That is, revenue equivalence fails. Also note that the outcome is no longer efficient since player 2—who has the smaller valuation—wins with positive probability.

17.3 Bounding on Social Welfare

Consider a general all-pay auction with n players under incomplete information. Define the social welfare function to be

$$W(b) = \sum_i u_i(b) + rev(b)$$

where $rev(b) = \sum_i b_i$ is the revenue to the seller.

Lemma 17.1 *Let b be a Nash equilibrium in an all-pay auction under incomplete information. Then*

$$E(W(b)) \geq \frac{1}{2} \max_i v_i.$$

It then follows immediately that the price of anarchy is bounded above by 2 since the optimal outcome occurs when the highest-valuation type wins the auction.

Proof: For ease of notation, order the players' valuations so that $v_1 \geq v_2 \geq \cdots \geq v_n$. Let b be a mixed-strategy Nash equilibrium. Since b is a Nash equilibrium, it must be true that for any other strategy b^* and for any player i,

$$Eu_i(b) \geq E(u_i(b_i^*, b_{-i})).$$

Let b^* be as follows: $b_i^* = 0$ if $i > 1$ and let player 1 consider a random bid according to the uniform distribution $b_1^* \sim U[0, v_1]$. That is, all players except player 1 (who has the highest valuation) bid 0, and player 1 bids randomly according to a uniform $[0, 1]$ distribution.

Now consider this b^* and any fix bid vector b. We have that for all $i > 1$, $u_i(b_i^*, b_{-i}) = 0$. Then, summing over all players,

$$E\left(\sum_i u_i(b)\right) \geq E\left(\sum_i u_i(b_i^*, b_{-i})\right) = 0 + E(u_1(b_1^*, b_{-1})). \tag{17.1}$$

Fix b_{-1} and denote $b_{\max} \equiv \max_{i \neq 1} b_i$. Then, since $b_1^* \sim U[0, 1]$, the probability that player 1 wins is

$$P(b_1^* > b_{\max}) = \frac{v_1 - b_{\max}}{v_1}.$$

Since $Eb_1^* = \frac{v_1}{2}$,

$$u_1(b_1^*, b_{-1}) = v_1 \frac{v_1 - b_{\max}}{v_1} - \frac{v_1}{2} = \frac{v_1}{2} - b_{\max} \geq \frac{v_1}{2} - rev(b)$$

where $rev(b) = \sum_i b_i \geq b_{\max}$. Since this holds for all b_{-1}, it holds in expectation. Continuing from (17.1),

$$E\left(\sum_i u_i(b)\right) \geq \frac{v_1}{2} - E(rev(b)).$$

Note that given our ordering of valuations, $v_1 = \max_i v_i$. Rearranging gives the desired inequality. ■