3.1 Price of Anarchy in Routing Games

3.1.1 Review of Last Class

We began with quick review of notation last class (can be found at http://www.cs.cornell.edu/courses/cs6840/2017sp/lecnotes/lec02.pdf)

Last class, we used our favorite example.

In this diagram, the Nash equilibrium had all travelers take the top edge. The optimal solution had half of the travelers on each edge. Thus, the Price of Anarchy was 4/3.

3.1.2 A More Difficult Routing Example

We redo the example but make it a little worse.

In this graph, the Nash equilibrium has all travelers take the top route and the cost for each traveler is 1. Thus, if $f$ is the global flow in Nash equilibrium, then the cost ($c(f)$) would be:

$$c(f) = 1$$

For the socially optimal solution, we will let $\epsilon$ travelers take the bottom edge and $1 - \epsilon$ take the top edge. If $f^*$ is the socially optimal flow, then the cost would be

$$c(f^*) = (1 - \epsilon)(1 - \epsilon)^N + \epsilon(1)$$

With the above equation, we see that as $N$ goes to infinity, the cost becomes $\epsilon$ (because the first term vanishes). Thus, as $N$ approaches infinity,

$$\text{Price of Anarchy} = \frac{c(f)}{c(f^*)}$$
the Price of Anarchy will also approach infinity.

### 3.1.3 Theorem and Proofs

Now let’s redraw the graph to be

![Graph](image)

Let \( r \) be any rate, \( x \) be the fraction of travelers on the top path, and \( r - x \) be the fraction on the bottom. The top path has cost \( c(x) \), and the bottom has a constant cost \( c(r) \).

In a Nash Equilibrium, all travelers will take the top path, so the cost of flow is

\[
\tilde{c}(f) = rc(r)
\]

In any other flow \( f^* \) where \( x \neq r \), then cost would be

\[
\tilde{c}(f^*) = xc(x) + (r - x)c(r)
\]

Let \( C \) be a set of monotone and continuous cost functions greater than 0. Define \( \alpha(C) \) to be

\[
\alpha(C) = \sup_{c \in C, \ 0 \leq x \leq r} \frac{rc(r)}{xc(x) + (r - x)c(r)}
\]  
(3.1)

The numerator in the fraction is the cost of flow in Nash Equilibrium and the denominator is the cost of flow of any other solution (where \( x \) fraction of travelers take the top path and \( r - x \) take the bottom).

Then the Price of Anarchy on the two-link graph is equal to \( \alpha(C) \). This is because the socially optimal solution cost in the denominator would maximize the fraction in \( \alpha(C) \) for a given cost function.

**Note:** The textbook is different in that it does not assume that the cost function is monotone and that it assumes \( x, r \geq 0 \), instead of \( 0 \leq x \leq r \). When calculating the Price of Anarchy, both equations yield the same answer. This is because when \( x > r \), \( \frac{rc(r)}{xc(x) + (r - x)c(r)} < 1 \) and is therefore, not the supremum.

To prove this, we will rearrange the ratio:

\[
\frac{rc(r)}{xc(x) + (r - x)c(r)} = \frac{rc(r)}{rc(r) + x(c(x) - c(r))}
\]

If \( x > r \), we see that the RHS is \( \leq 1 \) because \( c \) is a monotone function, so the denominator is greater than the numerator.

**Theorem 3.1** If \( C \) is a set of monotone and continuous cost functions greater than 0, then the Price of Anarchy on any network with \( c \in C \) is less than or equal to \( \alpha(C) \), the Price of Anarchy on the two link graph. In other words,

\[
\text{Price of Anarchy} \leq \alpha(C)
\]

To prove this, we have to first prove 2 claims:
Claim 3.2  \[
\sum_e f^*(e)c_e(f(e)) \geq \sum_e c_e(f(e))f(e) \text{ where } f \text{ is the flow in Nash Equilibrium and } f^* \text{ is any other flow}
\]

In words, claim 3.2 is stating that given a flow \( f \) in Nash Equilibrium, \textbf{if we fix the edge costs to those in } \( f \) in that graph, the cost of any other flow \( f^* \) (using the fixed edge costs) will be greater than or equal to the cost of the Nash Equilibrium flow.

**Proof:** Imagine cost \( c_e(f(e)) \) is fixed for all edges \( e \). We can rearrange the LHS and RHS to define the cost in terms of total path cost instead of total edge cost.

**LHS**

\[
\sum_e f^*(e)c_e(f(e)) \\
= \sum_p f^*_p c_p(f) \quad \text{as } c_p(f) = \sum_{e \in P} c_e(f(e)) \\
= \sum_i \sum_{p \in s_i \rightarrow t_i} f^*_p c_p(f) \quad \text{summing over all paths } p \text{ is equivalent to summing over all paths for each source-sink pair } i \\
\geq \sum_i \sum_{p \in s_i \rightarrow t_i} f^*_p c_i(f) \quad \text{ } c_i(f) \leq c_p(f) \text{ b.c. all travelers in Nash Equilibrium are using paths of lowest cost } c_i(f) \\
= \sum_i (c_i(f) \sum_{p \in s_i \rightarrow t_i} f^*_p) \quad \text{ } c_i(f) \text{ is not path dependent so can be factored out} \\
= \sum_i c_i(f)r_i \quad \text{as } r_i = \sum_{p \in s_i \rightarrow t_i} f^*_p \text{ (sum of flow of paths from } s_i \rightarrow t_i \text{ is equivalent to } r_i) \\
\]

**RHS**

\[
\sum_p f_p c_p(f) \\
= \sum_i c_i(f)r_i \\
\]

Because the final values in the rearranged LHS and RHS are equivalent and the LHS is greater than this value, then

\[
\sum_e f^*(e)c_e(f(e)) \geq \sum_e c_e(f(e))f(e)
\]

\[
\square
\]

Claim 3.3

\[
c(f) \leq \alpha(C)c(f^*) \tag{3.2}
\]

Claim 3.3 states that the cost of a Nash Equilibrium flow \( f \) is less than or equal to the product of \( \alpha(C) \) (defined as Equation 3.1) and the cost of flow \( f^* \) of any solution.

**Proof:** Recall that we defined \( \alpha(C) \) as

\[
\alpha(C) = \sup_{c \in C, 0 \leq x \leq r} \frac{rc(r)}{xc(x) + (r - x)c(r)}
\]
Because $\alpha(C)$ is a supremum, then we also know that

$$\alpha(C) \geq \frac{rc(r)}{xc(x) + (r - x)c(r)}$$

For the proof, we will set for each edge $e$: $r = f(e), x = f^*(e)$. Then the inequality becomes

$$\alpha(C) \geq \frac{f(e)c_e(f(e))}{f^*(e)c_e(f^*(e)) + (f(e) - f^*(e))c_e(f(e))}$$

Multiplying both sides by the denominator and summing over $e$ yields

$$\sum_e f(e)c_e(f(e)) \leq \alpha(C)(\sum_e f^*(e)c_e(f^*(e)) + \sum_e (f(e) - f^*(e))c_e(f(e)))$$

Recall that global cost $c(f) = \sum_e f(e)c_e(f(e))$, so we can replace these values.

$$c(f) \leq \alpha(C)(c(f^*) + \sum_e (f(e) - f^*(e))c_e(f(e)))$$

By Claim 3.2, $\sum_e (f(e) - f^*(e))c_e(f(e))) \leq 0$, so we can remove it from the RHS of the inequality. Therefore,

$$c(f) \leq \alpha(C)c(f^*)$$

\[\square\]

**Proof:**

To prove Theorem 3.1, we will use Claim 3.3 which states $c(f) \leq \alpha(C)c(f^*)$.

Dividing both sides by $c(f^*)$ yields the equation

$$\frac{c(f)}{c(f^*)} \leq \alpha(C)$$

$f^*$ was defined to be any flow, which includes the socially optimal flow. If $f^*$ were the socially optimal flow, then $\frac{c(f)}{c(f^*)}$ would be the Price of Anarchy. Therefore,

Price of Anarchy $\leq \alpha(C)$

\[\square\]