CS6840 - Algorithmic Game Theory (4 pages) Spring 2014

April 7 - Auction, Smoothness, and Second Price

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1 Outline

In the last few lectures we looked at smoothness analysis (but not quite exactly the "Roughgarden smoothness" for utility games) in the following examples of auctions:

- 2nd price item auction
- generalized second price
- greedy algorithms as mechanism

Today, we look at the general smoothness for an auction on many items, all sold on second price. Player i's value for item j is v_{ij}. All players have unit demands, so there is free disposal; if player i gets a set of items S, the value for that player is just the maximum valued item in that set, $v_i(S) = \max_{j \in S} v_{ij}$.

On Wednesday, we will look at a more general class of valuation, and that will tie up our study of auctions.

2 Smoothness and PoA

In the proofs of price of anarchy for both Generalized Second Price and mechanisms based on greedy algorithms, we made use of the following smoothness property ($\star$):

$$\exists \text{ bid } b_i^* \ \forall i \text{ s.t. } \forall b$$
$$\sum_i u_i(b_i^*, b_{-i}) \geq \lambda \cdot \text{OPT} - \mu \cdot \sum_i b_i(A_i)$$

Here, $\text{OPT} = \max_{\mathcal{O}} \sum_i v_i(\mathcal{O}_i)$ where $\mathcal{O} = (\mathcal{O}_1, ..., \mathcal{O}_n)$ is an allocation of items to players. Similarly, $A = (A_1, ..., A_n)$ is the allocation made by the mechanism on bids b.

For GSP, $\lambda = \frac{1}{2}, \mu = 1$; for mechanisms based on greedy algorithms, $\lambda = 1$, $\mu = c$ (the approximation factor).

We have shown this multiple times in different contexts, but as a review, the above lemma implies the following bound on PoA given some additional assumptions:

Claim 1. In a full information game, if $(\star)$ holds and $b_i(X) \leq v_i(X)\ \forall i\ \forall X$ (i.e. the bidders are *conservative*), then SW(CCE) $\geq \frac{\lambda}{\mu+1} \cdot$OPT

Proof. Recall that for a CCE (or a learning outcome) that is some distribution on bid b, we have

$$\mathbb{E}_b(u_i(b_i', b_{-i})) \leq \mathbb{E}_b(u_i(b)) \quad \forall \text{ player } i, \quad \forall \text{ alternate bid } b_i'$$

Hence, we have

$$\begin{aligned}
\mathrm{SW}(b) &\geq \mathbb{E}_b\left[\sum_i u_i(b)\right] && (1)\\
&\geq \mathbb{E}_b\left[\sum_i u_i(b_i^*, b_{-i})\right] && (2)\\
&\geq \lambda \cdot \mathrm{OPT} - \mu \cdot \mathbb{E}_b\left[\sum_i b_i(A_i)\right] && (3)\\
&\geq \lambda \cdot \mathrm{OPT} - \mu \cdot \mathbb{E}_b\left[\sum_i v_i(A_i)\right] && (4)\\
& && (5)
\end{aligned}$$

(1) holds since social welfare is the sum of utilities of all players plus the auctioneer utility
(2) holds because the distribution on b is a CCE
(3) is due to smoothness and linearity of expectations
(4) uses the conservative assumption

The right most term is just SW(b), so after rearranging we get the desired PoA bound. □

<u>Observation</u> With the conservative assumption, this is exactly Roughgarden's smoothness for utility games.

3 Auction example

Now, we look at the case of many items, second price auction with unit demand bidders. Recall that if we instead used first price auction, then the cost of the optimal social welfare is given by the maximum matching between players and items, i.e.

$$\text{OPT} = \max_{\text{matching } \mathcal{M}} \sum_{(i,j)\in\mathcal{M}} v_{ij}$$

Claim 2. second price item auction is (1,1) smooth in the sense of ($\star$).

Proof. We need to come up with some special bids b^*. Suppose j_i^* is the item player i gets in the optimal allocation. Then, let

$$\text{b}_i^* = \begin{cases} v_{ij} & \text{if } j = j_i^* \\ 0 & \text{otherwise} \end{cases}$$

Of course, the players don't know what j_i^* is so they can't bid like above practically. We will come back to address this issue.

We can lower bound the utility of a player i bidding b_i^* as

$$\begin{aligned} u_i(b_i^*, b_{-i}) &\geq v_{ij_i^*} - \max_{k\neq i} b_{kj_i^*} \\ &\geq v_{ij_i^*} - \max_{k} b_{kj_i^*} \end{aligned}$$

Summing over all players,

$$\begin{aligned} \sum_i (b_i^*, b_{-i}) &\geq \sum_i v_{ij_i^*} - \sum_i \max_k b_{kj_i^*} \\ &= \text{OPT} - \sum_i b_i(A_i) \end{aligned}$$

The last equality follows from observing that since $\max\limits_k b_{kj_i^*}$ is the maximum bid in b for item j_i^*, if we sum over all players, we are effectively summing the highest bid over all items, which is equal to $\sum\limits_i b_i(A_i)$.

□

4 Learning and PoA bound

We pointed out above that the players do not actually know j_i^*, so though we were able to prove the claim we may wonder what the claim actually means practically. The idea is that we let players use learning, where their options are between the n items; then they bid v_{ij} for the chosen item j and 0 for all others.

The corollary of Claim 2 is that if the players use learning, social welfare in expectation is at least $\frac{1}{2}$OPT in the above setting.

Now, is conservativeness a reasonable assumption? This is not necessarily so, as the following example shows: consider a game with two items A and B. Player 1 has value 1 for both items, and player 2 has value $\frac{1}{2}$ for both items. We may assume there are other players with lower values. Player 1 bids 1 for one item and 0 for the other. Player 2 bids $\frac{1}{2}$ for each item, since this is a full information and he knows that he's going to lose one item to player A. Now, player 2 is *not* conservative as $b_2(A, B) = \frac{1}{2} + \frac{1}{2} = 1 > v_2(A, B) = \frac{1}{2}$.