Algorithmic Game Theory	**Problem Set 4**
CS 6840 Spring 2014	**Due Monday, April 21st**

There are 5 questions on this problem set of varying difficulty. For full credit you should solve 4 of the 5 problems. Solving all 5 results in extra credit. A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck.

You may work in pairs and hand in a shared homework with both of your names marked. You may discuss homework questions with other students, but closely collaborate only with your partner. You may use any fact we proved in class without proving the proof or reference, and may read the relevant chapters of the book. However, you may **not use other published papers, or the Web to find your answer**.

Solutions can be submitted on CMS in pdf format (only). If you have a partner, write both names on the solution, but only upload or submit it once. In any case, please type your solution or write neatly to make it easier to read. If your solution is complex, say more than about half a page, please include a 3-line summary to help us understand the argument.

(1) Consider the Generalized Second Price auction considered in lecture March 24-26. Recall that each slot i has a "click-through-rate" α_i, and each ad j has a "quality factor" γ_j such that if ad j is placed on slot i it gets a click with probability $\alpha_i\gamma_j$. We also assume that the advertiser has a private value v_i for each click. Now assume that a small set of advertisers are competing for placing their ads. We can consider this a full information game. At a Nash equilibrium of this game, the advertisers know each-other's bidding strategy, and each is best-responding to the strategies of the other players. However, for this problem we model the "quality factor" γ_j as random. Assume the platform (Google) correctly computes γ_j for each advertiser j, but the advertisers have to place their bids without knowing their quality factors (or the quality factors of other advertisers); instead they only know the distribution of quality factors. Assume that the vectors of quality factors γ is pulled from a known joint distributions Γ. Note that values γ_j for different advertisers may be correlated. In this game, the advertisers place bids b_j, the platform computes γ_j for all j, and uses the GSP mechanism, as started in lecture on March 24. Assume that players are bidding to maximize their expected utility (i.e., are risk-neutral).

(a) Give the definition of Nash equilibrium for this game.
(b) Show that this game is $(1/2, 1)$-smooth using bids (instead of prices).
(c) Show that the expected social welfare at a Nash equilibrium of this game is at least 1/4th of the expected value of the maximum possible welfare assuming all bids $b_j \leq v_j$.
(d) Now assume that the players do not reach a Nash equilibrium, instead they all employ a no-regret learning algorithm to optimize their bids. We'll use $u_i(\gamma, b)$ to be ith player's expected utility in the GSP outcome with bids b and quality factors γ. Assume that over a sequence of T steps, each advertiser i uses a sequence of bidding strategies that guarantees no-regret, that is,

$$\sum_t E_\gamma(u_i(\gamma, b^t)) \geq \sum_t E_\gamma(u_i(\gamma, b'_i, b^t_{-i})$$

for any fixed bid b'_i. Show that if $b^t_i \leq v_i$ for all i and all t, than the expected social welfare is at least 1/4 of the maximum possible expected welfare.

(2) Assume there are n identical items on sale, and each buyer has a concave valuation function for receiving multiple items. We'll use $v_i(k)$ to denote player i's value for k items $v_i(k)$, and will assume that $v_i(k)$ is a monotone nondecreasing and concave function of k, and assume that $v_i(0) = 0$ for all i. A standard auction format (called uniform price auction) works as follows. We say that player i's marginal value for the kth item is $m_i(k) = v_i(k) - v_i(k-1)$. Note that our assumptions imply that $m_i(k)$ is nonnegative, and monotone decreasing in k. Each player is asked to submit "claimed marginal values" $b_i(k)$ for all k, that needs to be non-negative, and non-increasing in k. Each items is assigned in order of marginal values to the users. So the first item goes to the person with highest $b_i(1)$, after some items are assigned, we take from each player the claimed marginal value for his/her next item, and assign the item to the person with highest such claim. At the end of the process all users pay a uniform price for the items they won: the claimed marginal value of the last assigned item.

For example, if there are two players with values $v_1(1) = 5$, $v_1(2) = 8$, and $v_1(3) = 9$, and $v_2(1) = 2$, $v_2(2) = 4$, and $v_2(3) = 4$, then the marginal values are $m_1(1) = 5$, $m_1(2) = 3$, and $m_1(3) = 1$, and $m_2(1) = m_2(2) = 2$ and $m_2(3) = 0$. If the auction had 4 items on sale, and both bidders reported their true valuations, they would get 2 items each and would get charges 2 for each item.

(a) Is this auction truthful? i.e., is reporting the player's true marginal value always a Nash equilibrum? explain your answer.
(b) A variant of this auction uses bidding and the same allocation, but charges for each allocated item the highest claimed marginal value (over all players and requests) for which no item was allocated. In the example above with truthful bidding, this results in the same allocation but price 1 for each item. Is this auction truthful?
(c) Show that the original game when we charge all buyers the reported marginal cost of the last allocated item is a $(1/2, 1)$-smooth game (using the version with bids and not with prices, and hence has a price of anarchy of at most 4 assuming that the bidders are conservative, and never bid above their marginal value).
(d) Extend the price of anarchy bound to a Bayesian version of the game, where valuation functions v come from a joint distributions $\mathcal{F}$, and players bid to maximize their expected utility.

(3) In the lecture on Friday, March 28th we have shown that if a greedy algorithm is a c-approximation algorithm, then the mechanism using this algorithm with critical value pricing has a $c + 1$ price of anarchy, i.e., the social welfare at a Nash equilibrium is at least a $c + 1$ fraction of the maximum possible welfare. Show that under the same conditions, the price of anarchy of greedy mechanism with the first price rule is bounded by c. (First price rule here means that a bidder i with allocated set A gets to pay $b_i(A)$.)

(4) Consider an auction for an item that is reproducible for free (such as music). Assume each bidder has a value v_i for the item, we have unlimited supply, but the seller incurs at fixed cost 1, if there are any buyers. We say that an auction is *budget balanced* if whenever any buyers receive an item, the total price charged is 1.

(a) with n buyers, consider this as a game with $n+1$ players where the seller is also a player with value -1 for having to sell. The VCG mechanism is designed to maximize social welfare. Give a simple description of the mechanism for this case. Prove that it is not budget balanced.

Will the outcome guaranteed to make too much or too little money? (i.e., is it guaranteed to collect at least 1 when items are allocated? is it guarantee to collect at most 1 when items are allocated.

(b) consider the following simple mechanism. Start with S the set of all n players, and offer price $p = 1/|S|$ to each. Offers buyers an opportunity to drop out (buyers with value $v_i < p$ may want to drop out at this point). Let S be the set of remaining buyers. Offer this group of buyers the price $p = 1/|S|$; repeat till either S is empty, or till the process stabilizes. Clearly this process is budget balanced. Is truthful bidding dominant strategy? i.e., can it ever be advantageous for a bidder to stay in the auction when $p > v_i$?

(c) This mechanism doesn't maximize social welfare. For example, if player i has value just below $1/i$, then in n iteration of this process all bidders drop out, despite the fact that welfare can be quite high by selling to all agents. Show that this is the maximum possible difference between the achievable social welfare and the welfare obtained by the mechanism.

(5) We have seen in lecture on April 7th that if users have unit demand, and are bidding conservatively then in the game of independent second price item auction coarse correlated equilibria equilibria have welfare at least 1/2 of the maximum possible. This question aims to explore if something similar can also be said about the analogous Bayesian game in which types (valuations) of players are random with known and independent distributions. In a Nash equilibrium of this Bayesian game the player's strategy can only depend on his own type. Here we will consider only pure equilibria, i.e., assume that the player's strategy is a deterministic function of his/her type, and use $b_j^v(a)$ to denote player j's bid for item a when her type is v_j. Let A_i^v denote the allocation that player i gets at Nash and let O_i^v denote the optimum allocation for player i when the player types are v. Also let $p_{-i}^{v_{-i}}(a) = \max_{j \neq i} b_j^v(a)$ the maximum bid of all players except i for item a and type profile v at Nash. Note that $p_{-i}^{v_{-i}}(a)$ is a random variable that depends on the valuations v_{-i} (and the resulting bids). Also, define $b^v(a) = \max_j b_j^v(a)$, the highest bid for item a at Nash. Let w be a random valuation vector with coordinates selected at random according to the Bayesian distribution of valuations.

(a) Consider a fixed valuation v_i for a player i. Show that we have

$$E_{w_{-i}}(v_i(O_i^{(v_i,w_{-i})})) - E_{v_{-i},w_{-i}}(\sum_{a \in O_i^{(v_i,w_{-i})}} p_{-i}^{v_{-i}}(a)) \leq E_{v_{-i}}(v_i(A_i^v)).$$

(b) Consider the term $E_{v_{-i},w_{-i}}(\sum_{a \in O_i^{(v_i,w_{-i})}} p_{-i}^{v_{-i}}(a))$, take expectation also over v_i and w_i, and take the sum over i. Show that the expectation is bounded by $E_v(\sum_i v_i(A_i^v))$ under the conservative assumption. (Hint: Useful to think of the contribution to this expectation by the terms $\sum_{a \in O_i^x} p_{-i}^{v_{-i}}(a)$ for a fixed vector $x = (v_i, w_{-i})$.)

(c) Consider the term $E_{w_{-i}}(v_i(O_i^{v_i,w_{-i}}))$, take expectation also over v and w_i, and sure for all values of i. Show that the expectation of this sum is the same as $E_v(\sum_i v_i(O_i^v))$.

(d) Show that the expected social welfare in a pure Nash equilibrium in this Bayesian game is at least 1/2 of the optimum.